Edge/Vector Finite Elements Method for Maxwell's Equation

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Model problem

Given a bounded Lipschitz polyhedron in $\mathbb{R}^3$ with connected boundary Γ and unit outward normal n, we seek non-trivial Electric fields $\mathbf{E} \equiv (E_1(x), E_2(x), E_3(x))$ which satisfies

$$\nabla \times (\nabla \times \mathbf{E}) + c(x)\mathbf{E} = \mathbf{f}(x), \quad \text{in } \Omega,$$
$$n \times \mathbf{E} = 0 \quad \text{on } \Gamma.$$

Where $c(x)$ is assumed to be a bounded in Ω and $\mathbf{f} \in L_2(\Omega)$. For simplicity, we set $c(x) \equiv 1$ for this lecture.

Model problem

- The model problem describes how electric fields behave inside a material with sources $\mathbf{f}$ and spatially varying properties $c(x)$.
- The homogeneous Dirichlet boundary condition $n \times \mathbf{E} = 0$ is referred to as a perfectly conducting boundary condition.
- This boundary condition ensures that the electric field vanishes tangentially on the surface, effectively modeling a perfectly conducting enclosure.

Solution space

Definition

$H(\text{curl}, \Omega) = \{\mathbf{u} \in L_2(\Omega) : \nabla \times \mathbf{u} \in L_2(\Omega)\}$

Exercise

Show that the space $H(\mathrm{curl}, \Omega)$, endowed with the inner product

$$(\mathbf{v}, \mathbf{u})_{H(\mathrm{curl};\Omega)} = (\mathbf{v}, \mathbf{u})_{L_2(\Omega)} + (\nabla \times \mathbf{v}, \nabla \times \mathbf{u})_{L_2(\Omega)}.$$

is a Hilbert space.

- The norm induced by the inner product $(\mathbf{v}, \mathbf{u})_{H(\text{curl},\Omega)}$ is

$$\|\mathbf{u}\|_{H(\text{curl};\Omega)} = \left(\|\mathbf{u}\|^2_{L_2(\Omega)} + \|\nabla \times \mathbf{u}\|^2_{L_2(\Omega)}\right)^{\frac{1}{2}}.$$

Trace theorem

Trace theorem

Let n denote the outward normal to the boundary Γ. The map

$$\gamma_t : \mathbf{u} \to \mathbf{u} \times n$$

is continuous and linear from $H(\text{curl}, \Omega)$ to $\left(H^{-\frac{1}{2}}(\Gamma)\right)^3$.

Definition

$H_0(\text{curl}, \Omega) = \{\mathbf{u} \in H(\text{curl}, \Omega) : \gamma_t(u) = 0 (n \times \mathbf{u} = 0) \text{ on } \Gamma\}$

Green's formula

Let $\mathbf{u}$ be in $H(\text{curl}, \Omega)$ and $\mathbf{v}$ be a test function in $H^1(\Omega)$. We then have

$$\int_\Omega \mathbf{v} \cdot (\nabla \times \mathbf{u})\, dx = \int_\Omega \mathbf{u} \cdot (\nabla \times \mathbf{v})\, dx + \int_\Gamma (n \times \mathbf{u}) \cdot \mathbf{v} ds.$$

Variational formulation

The variational formulation is obtained by multiplying the model equation with a test function and integrating over Ω:

$$\int_\Omega [\nabla \times (\nabla \times \mathbf{E})] \cdot \mathbf{v} \, d\Omega + \int_\Omega \mathbf{E} \cdot \mathbf{v} \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\Omega.$$

Applying integration by parts/Green's formula, we get

$$\int_\Omega (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{v}) \, d\Omega + \int_\Omega \mathbf{E} \cdot \mathbf{v} \, d\Omega = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\Omega, \quad (1)$$

for all $\mathbf{v} \in H_0(\mathrm{curl}, \Omega)$.

Bilinear form

The bilinear form defined on $H_0(\mathrm{curl}, \Omega) \times H_0(\mathrm{curl}, \Omega)$ by

$$a(\mathbf{E}, \mathbf{v}) = \int_\Omega (\nabla \times \mathbf{E}) \cdot (\nabla \times \mathbf{v})\, d\Omega + \int_\Omega \mathbf{E} \cdot \mathbf{v}\, d\Omega.$$

Now, the problem is reduced to find $\mathbf{E} \in H_0(\mathrm{curl}, \Omega)$ such that

$$a(\mathbf{E}, \mathbf{v}) = \mathbf{f}(\mathbf{v}), \quad \forall \mathbf{v} \in H_0(\mathrm{curl}, \Omega). \tag{2}$$

Lemma (Boundedness)

There exists a positive constant $C > 0$ such that

$$|a(u,v)| \leq C\|u\|_{H(\mathrm{curl},\Omega)}\|v\|_{H(\mathrm{curl},\Omega)} \quad \forall u, v \in H_0(\mathrm{curl},\Omega).$$

Consider

$$\begin{aligned}
|a(u,v)| &\leq \left|\int_\Omega (\nabla \times u)\cdot(\nabla \times v)\,dx\right| + \left|\int_\Omega u \cdot v\,dx\right| \\
&\leq \|\nabla \times u\|_{L^2(\Omega)}\,\|\nabla \times v\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}\,\|v\|_{L^2(\Omega)} \\
&\leq \left(\|\nabla \times u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \left(\|\nabla \times v\|^2_{L^2(\Omega)} + \|v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\
&\leq \|u\|_{H(\mathrm{curl},\Omega)}\|v\|_{H(\mathrm{curl},\Omega)},
\end{aligned}$$

where $C = 1$.

Lemma (Coercive)

There exists a positive constant $\alpha > 0$ such that

$$|a(u, u)| \geq \alpha \|u\|^2_{H(\text{curl},\Omega)} \quad \forall u \in H_0(\text{curl}, \Omega).$$

Lax-Milgram

Suppose $a : X \times X \to \mathbb{R}$ is a bounded and coercive bilinear form. Then for each $f \in X'$, there exists a unique solution $u \in X$ to

$$a(u, v) = f(v) \quad \forall v \in X,$$

and

$$\|u\|_X \leq \frac{C}{\alpha}\|f\|_{X'},$$

where C and α are the constants in the boundedness and coercivity definitions above.

- Therefore, there exists a unique solution $\mathbf{E} \in H_0(\text{curl}, \Omega)$ to

$$a(\mathbf{E}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in H_0(\text{curl}, \Omega),$$

and

$$\|\mathbf{E}\|_{H(\text{curl},\Omega)} \leq \|\mathbf{f}\|_{L_2(\Omega)},$$

- Let $c(x) \equiv -\alpha$, $\alpha > 0$. The model problem

$$\nabla \times (\nabla \times \mathbf{E}) - \alpha \mathbf{E} = \mathbf{f}(x), \quad \text{in } \Omega,$$
$$n \times \mathbf{E} = 0, \quad \text{on } \Gamma.$$

  Then existence and uniqueness of **E** follows from the Fredholm Alternative.

- If $c(x) \equiv 0$, then model leads to Saddle point system

Edge element discretization

- Assume that the bounded domain Ω is partitioned into a regular mesh $\mathcal{T}_h$ consisting of triangles (in 2D) or tetrahedra (in 3D) with mesh size h.

Characterization of $H(\mathrm{curl}, \Omega)$

Let $K_-$and K_+be two polygonal (resp. polyhedral) Lipschitz domains in $\mathbb{R}^d$, with a common edge (resp. common edge or face) $e = \partial K_- \cap \partial K_+ \neq \emptyset$ and denote by $\Omega = K_- \cup K_+$their union. A function v is in $H(\mathrm{curl}; \Omega)$ if and only if the restricion $v_-$of v to $K_-$is in $H\,(\mathrm{curl}; K_-)$, the restricion $v_+$of v to K_+ is in $H\,(\mathrm{curl}; K_+)$and the tangential jump over e vanishes: $(v_- \times n_-) + (v_+ \times n_+) = 0$ on e.

- The Nedelec's elements space is defined as

$$V_h^k = \left\{ v \in H_0(\mathrm{curl}, \Omega) \ : \ v|_K \in \mathcal{R}_k \text{ for all } K \in \mathcal{T}_h \right\}$$

where

$$\mathcal{R}_k = (P_{k-1})^d \ \oplus \ \left\{ \mathbf{p} \in (\widetilde{P}_k)^d \ \middle| \ \mathbf{x} \cdot \mathbf{p} = 0 \right\},$$

P_k is the space of polynomials of total degree at most k, and $\widetilde{P}_k$ denotes the space of homogeneous polynomials of degree k.

- The discrete variational formulation corresponding to (2) is:
  Find $\mathbf{E}_h \in V_h^k$ such that

$$a(\mathbf{E}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \qquad \forall \, \mathbf{v}_h \in V_h^k,$$

  where $V_h^k \subset H_0(\mathrm{curl}, \Omega)$ denotes a finite-dimensional conforming subspace.

Finite element basis functions. A general definition of finite elements on an arbitrary polyhedra is given by the following.

Definition

A finite element (K, P, A), consists of

- K, a polyhedral domain;
- P , a vector space of polynomials defined on K having a basis $\{\phi_1, \phi_2, ..., \phi_N\}$ (called shape functions);
- A, a set of linear functionals defined on P having a basis $l_1, l_2, ..., l_N$ (called the degrees of freedom).

Edge Elements

- Let K denote a triangle (in 2D) or a tetrahedron (in 3D), specified by its set of vertices and edges
- For the k-th edge e_k with vertices (i, j) and the direction from i to j, the basis ϕ_k and corresponding degree of freedom $l_k(\cdot)$ are

$$\phi_k = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i, \qquad l_k(\mathbf{v}) = \int_{e_k} \mathbf{v} \cdot \mathbf{t}\, ds \approx \frac{1}{2}\left[\mathbf{v}(i) + \mathbf{v}(j)\right] \cdot \mathbf{e}_k,$$

  where the quadrature is exact when $\mathbf{v} \cdot \mathbf{t}$ is linear. And, λ_i are barycentric coordinates.
- The basis functions and degrees of freedom satisfy

$$l_k(\phi_m) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$$

- The local edge element space

$$NE^0 = \text{span}\{\phi_k, k = 1, 2, 3 (\text{for triangle}), k = 1, 2, 3..., 6 (\text{for tetrahedron})$$

- The global finite element space is obtained by assembling the local edge element spaces on all elements and identifying the degrees of freedom associated with the same geometric edges.

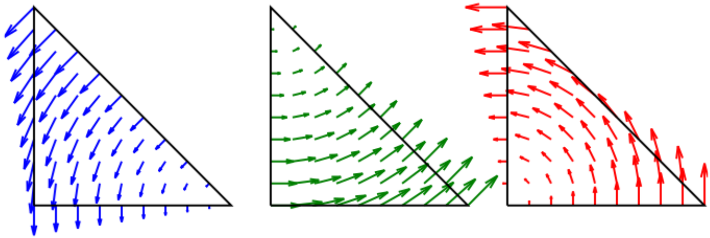

Figure: Basis of NE^0 in a triangle

- Vector field ϕ_k of edge k is orthogonal to other edges.

- The edge element space NE^0 is $H(\text{curl}, \Omega)$ conforming subspace.

Proof: To show the obtained spaces are indeed in $H(\text{curl}, \Omega)$, it suffices to verify the tangential continuity of the piecewise polynomials. Consider a tetrahedron T and a triangular face f. Let x_f be the vertex opposite to f and λ_f the barycentric coordinate corresponding to x_f.
On the face f, we have $\lambda_f|_f = 0$, and $\nabla\lambda_f$ is normal to f, so
$\nabla\lambda_f \times n_f = 0$.
For an edge e using x_f as a vertex, the Nédélec basis function is a linear combination of $\lambda_i \nabla\lambda_f$ and $\lambda_f \nabla\lambda_i$. Restricting to f:

$$\lambda_f \nabla\lambda_i = 0, \quad \nabla\lambda_f \times n_f = 0 \quad \Rightarrow \quad \phi_e|_f \times n_f = 0.$$

Therefore, for any $v \in NE^0(T)$, the tangential trace $v|_f \times n_f$ depends only on the basis functions of the edges of f. Since these match across neighboring tetrahedra, the tangential continuity is satisfied, proving that the space is a subspace of $H(\mathrm{curl}; \Omega)$.

Given a triangulation $\mathcal{T}_h$ with mesh size h. Define

$$I_h^{\mathrm{curl}} : V \cap \mathrm{dom}(I_h^{\mathrm{curl}}) \to NE^0(\mathcal{T}_h)$$

as follows: given a function $u \in V$, define $u^I = I_h^{\mathrm{curl}} u \in NE^0(\mathcal{T}_h)$ by matching the d.o.f.

$$l_e(I_h^{\mathrm{curl}} u) = l_e(u) \qquad \forall e \in \mathcal{E}_h(\mathcal{T}_h).$$

Namely,

$$u^I = \sum_{e \in \mathcal{E}_h} \left(\int_e u \cdot t \, ds \right) \phi_e.$$

Reference elements

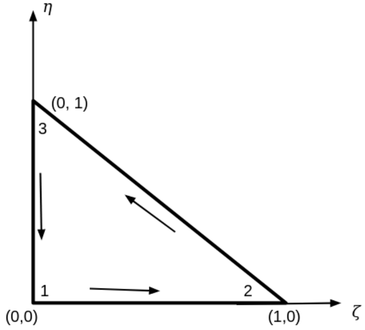

Figure: Reference triangle element.

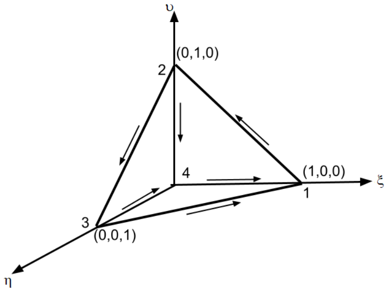

Figure: Reference tetrahedron element.

- The barycentric coordinates for the reference triangle

$$\lambda_1 = 1 - \zeta - \eta,$$
$$\lambda_2 = \zeta,$$
$$\lambda_3 = \eta.$$

- The edge basis functions for the triangle element are

$$\hat{\phi}_{12} = (1 - \eta, \zeta),$$
$$\hat{\phi}_{23} = (-\eta, \zeta),$$
$$\hat{\phi}_{31} = (-\eta, \zeta - 1).$$

- The edge basis functions for the triangle element are

$$\hat{\phi}_{41} = (\,1 - \nu - \eta,\ \zeta,\ \zeta\,),$$

$$\hat{\phi}_{12} = (-\nu,\ \zeta,\ 0),$$

$$\hat{\phi}_{24} = (-\nu,\ -1 + \zeta + \nu,\ -\nu),$$

$$\hat{\phi}_{23} = (0,\ -\eta,\ \nu),$$

$$\hat{\phi}_{34} = (-\eta,\ -\eta,\ -1 + \zeta + \nu),$$

$$\hat{\phi}_{31} = (\eta,\ 0,\ -\zeta).$$

Piola transformation

- An affine triangle or tetrahedron K is described by affine element map $F_K : \hat{K} \to K$ by

$$K \ni x = F_K(\hat{x}) = B_K \hat{x} + b_K.$$

- The shape function $\phi(x)$ on the element $K = F_K(\hat{K})$ are obtained from the reference shape functions by

$$\phi(x) = \left(\hat{D} F_K^{-T} \hat{\phi} \right) \circ F_K^{-1}(x),$$

where $\hat{D} F_K$ is the Jocobian $\frac{d}{d\hat{x}} F_K(\hat{x})$ of the element map.

Implementation

Suppose that the solution domain Ω is discretized using polygonal elements. By interpolating the vector field $\mathbf{E}$ with the edge basis functions, we write

$$\mathbf{E}(x) = \sum_{i=1}^{N} E_i \, \phi_i(x),$$

where N is the number of edges in the element. Taking the test function $\mathbf{v} = \phi_i(x)$ for $i = 1, 2, \ldots, N$ leads to the following matrix equation:

$$[K^e]\,[E^e] = [S^e].$$

This represents the finite element equation at the element level.

- The coefficients of the element stiffness matrix and the load vector are computed using the following expressions:

$$K_{ij}^e = \int_{\Omega_e} (\nabla \times \phi_i) \cdot \left(\nabla \times \phi_j\right) \, dx + \int_{\Omega_e} \phi_i \cdot \phi_j \, dx,$$

$$S_i^e = \int_{\Omega_e} f \cdot \phi_i \, dx,$$

  for $i, j = 1, 2, \ldots, N$, where N is the number of edges in the element.
- By assembling the element-level matrices and vectors, the global linear system is obtained:

$$[K][E] = [S].$$

Advantages of edge elements

- Edge elements allow the normal component of a vector field to be discontinuous across element boundaries.
- Edge elements require fewer degrees of freedom compared to nodal elements.

Example

Consider $\Omega = [0,1] \times [0,1]$, $c(x) = 1$ for all $x \in \Omega$ and

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = (\cos(\pi x)\sin(\pi y), \sin(\pi x)\cos(\pi y)).$$

The analytical solution to the model problem is given by

$$\mathbf{E} = (\cos(\pi x)\sin(\pi y), \sin(\pi x)\cos(\pi y))$$

References

For more information:

- **Title:** *Finite Element Method for Maxwell's Equations*
- **Author:** Peter Monk
- **ISBN:** 978-0198508885
- **Publisher:** Oxford Science Publications, Oxford University Press