NUMERICAL SOLUTIONS OF ODES: CONVERGENCE AND STABILITY ANALYSIS

Dr. Jitendra Kumar

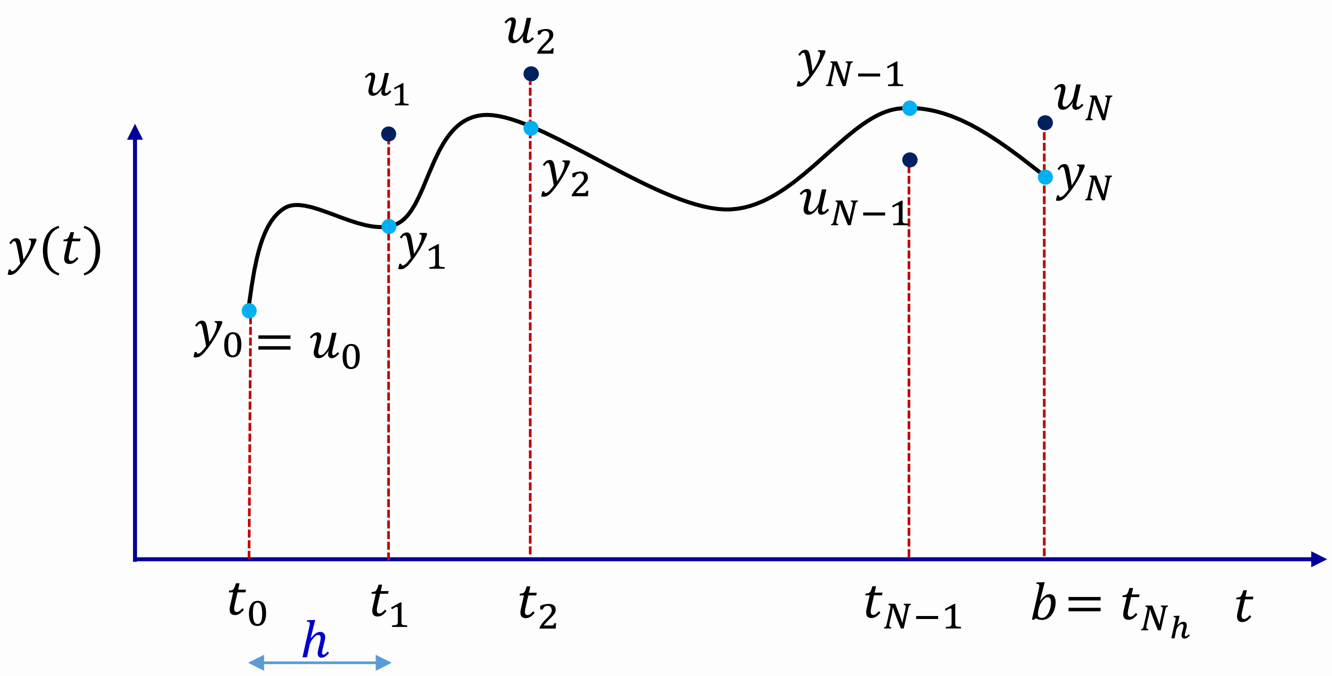
Professor
Department of Mathematics
Indian Institute of Technology ROPAR

Consider $\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0, t \in [t_0, b]$

Assume that the solution exists and is unique, i.e., f is uniformly *Lipschitz continuous w.r.t.* the second argument.

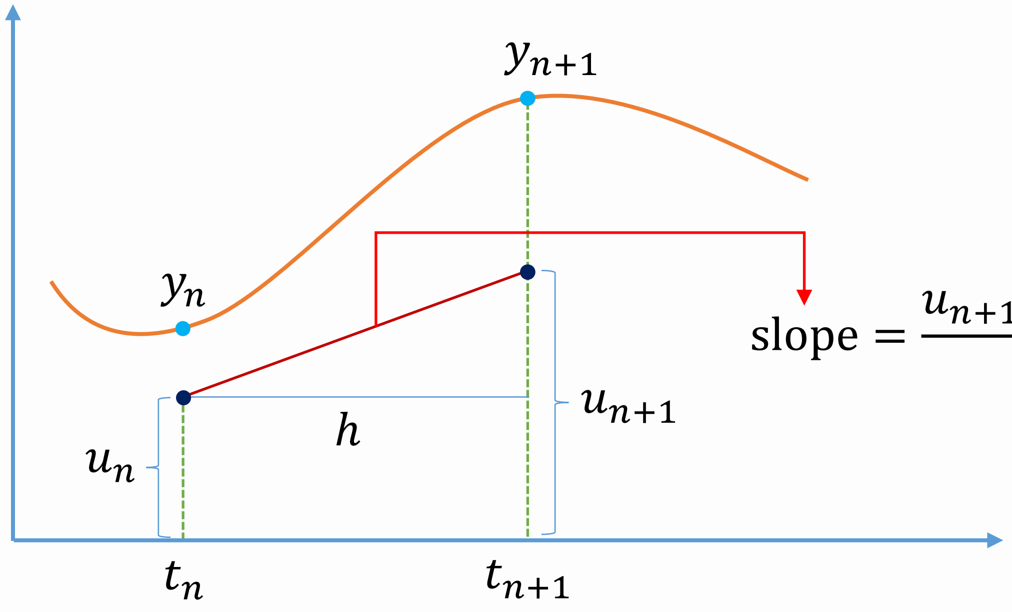
Numerical Solution:

$$u_n \approx y(t_n) = y_n$$



Key Idea of the Single Step Methods

Consider IVP: $\dfrac{dy}{dt} = f(t, y); \quad y(t_0) = y_0, t \in [t_0, b]$



$$\text{slope} = \frac{u_{n+1} - u_n}{h}$$

Numerical Method

$$\boxed{u_{n+1} = u_n + h \times \text{slope}}$$

$$u_{n+1} = u_n + h \times \phi(t_n, u_n, t_{n+1}, u_{n+1}, f, h)$$

How to get ϕ?

Different approximations of ϕ leads to different numerical methods.

Single Step Methods

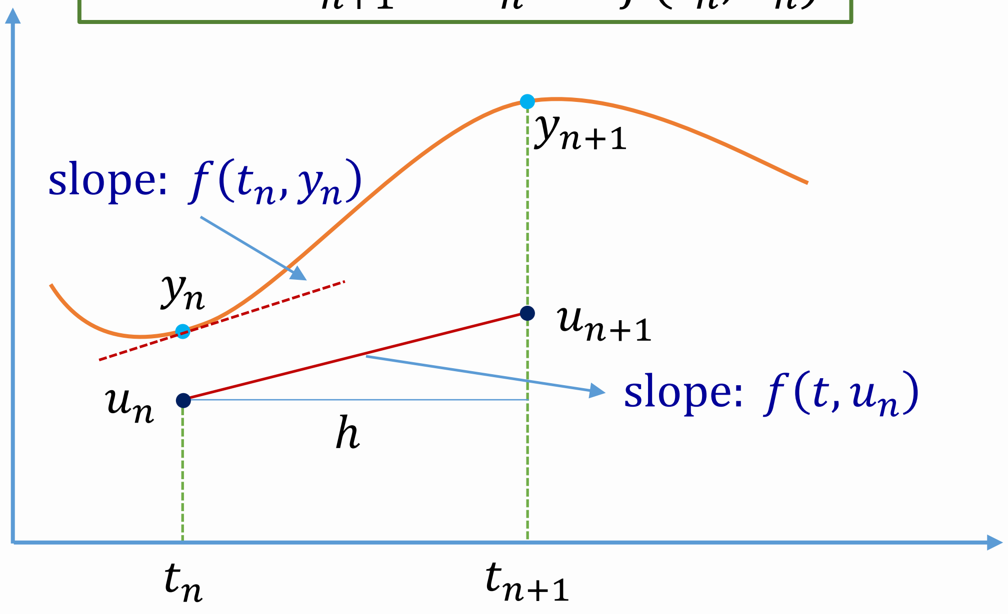
Explicit (ϕ is computed in terms of u_n)

$$u_{n+1} = u_n + h \times \phi(t_n, u_n, f, h)$$

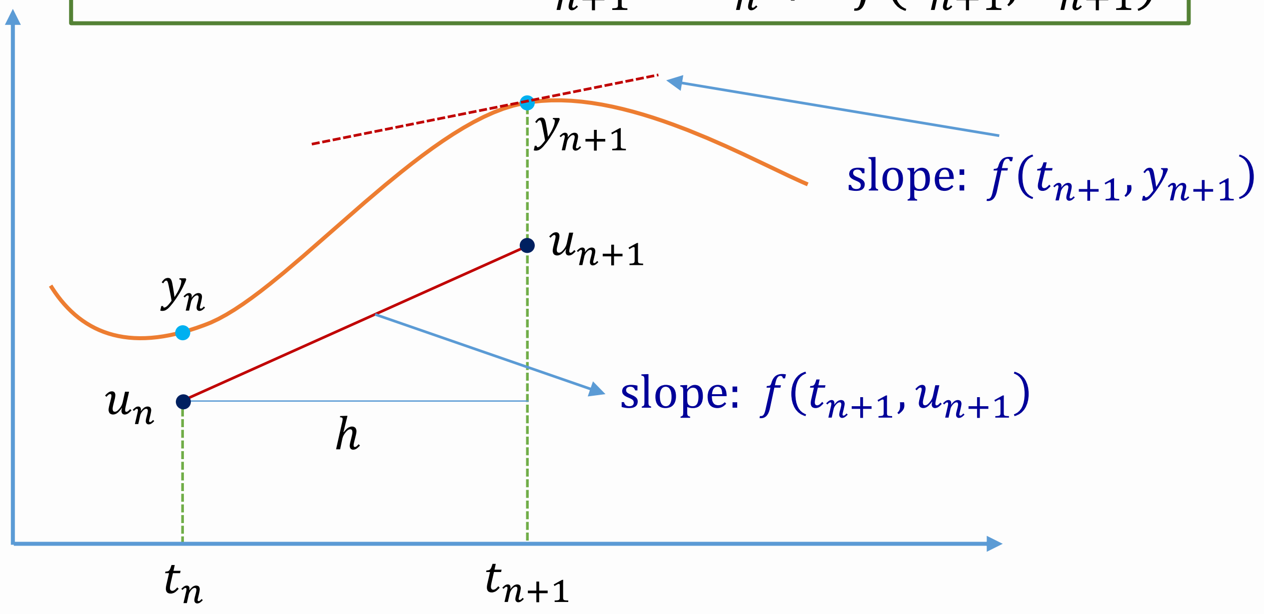
Implicit (ϕ depends on u_{n+1} itself through f)

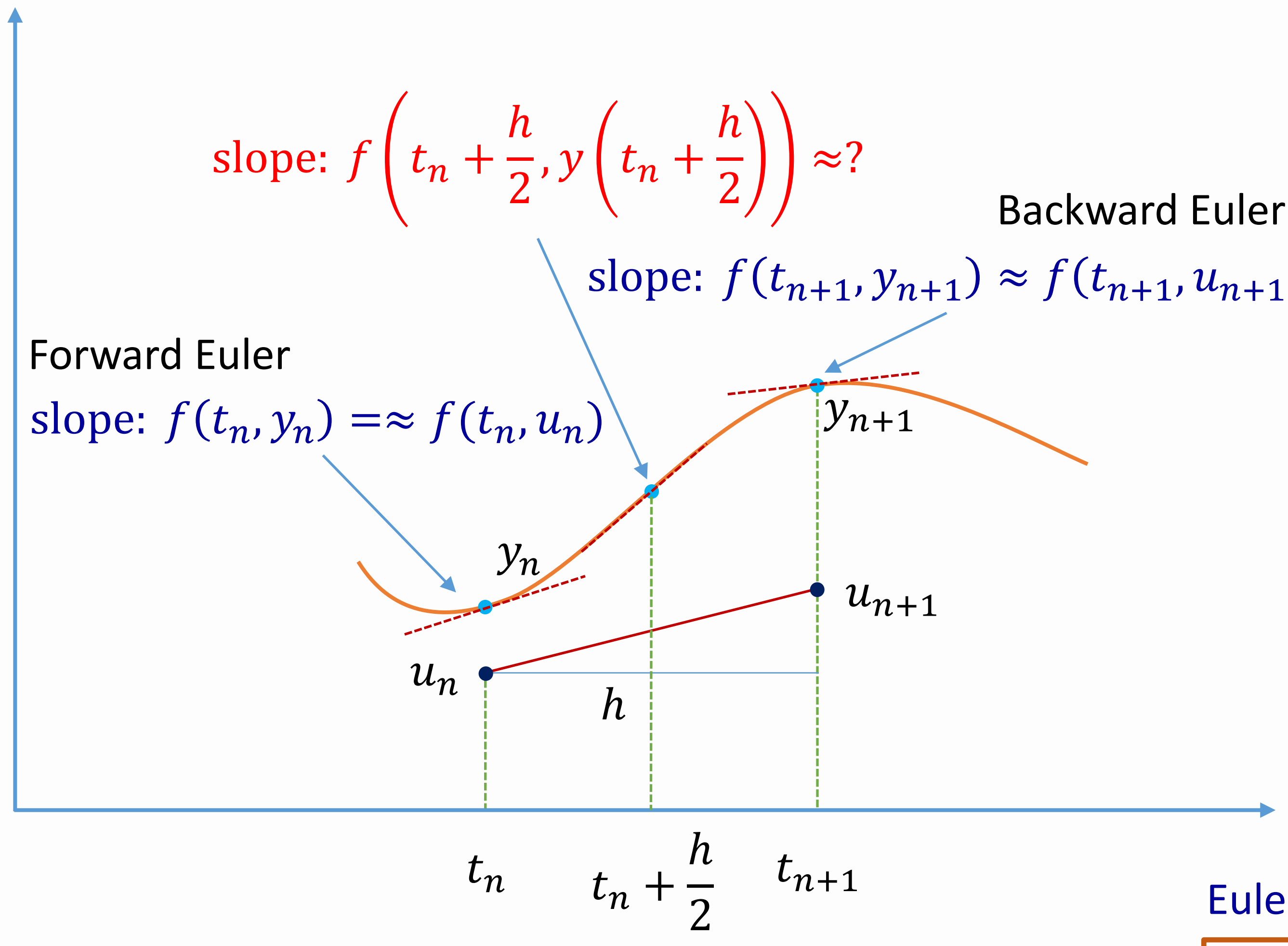
$$u_{n+1} = u_n + h \times \phi(t_n, u_n, t_{n+1}, u_{n+1}, f, h)$$

Euler : $u_{n+1} = u_n + hf(t_n, u_n)$



Backward Euler : $u_{n+1} = u_n + hf(t_{n+1}, u_{n+1})$





Explicit Midpoint Method

$$u_{n+1} = u_n + hf\left(t_n + \frac{h}{2}, u_n + \frac{h}{2} f_n\right)$$

Implicit Midpoint Method

$$u_{n+1} = u_n + hf\left(t_n + \frac{h}{2}, \frac{u_n + u_{n+1}}{2}\right)$$

Implicit Trapezoidal Method

$$u_{n+1} = u_n + \frac{h}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right)$$

Euler-Cauchy Method (Heun's Method) (Improved Euler)

$$u_{n+1} = u_n + \frac{h}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_n + hf(t_n, u_n))\right)$$

Runge-Kutta Methods

$$u_{j+1} = u_j + h(\text{weighted average of slopes on the given interval})$$

Explicit Methods (n-stage):

$$k_1 = f(t_j, u_j)$$

$$k_2 = f(t_j + c_2 h, u_j + h a_{21} k_1)$$

$$k_3 = f(t_j + c_3 h, u_j + h a_{31} k_1 + h a_{32} k_2)$$

$$\vdots$$

$$k_n = f(t_j + c_n h, u_j + h a_{n1} k_1 + h a_{n2} k_2 + \cdots + h a_{nn-1} k_{n-1})$$

$$u_{j+1} = u_j + h[w_1 k_1 + w_2 k_2 + \cdots + w_n k_n]$$

Euler-Cauchy Method (Heun's Method)

$$u_{j+1} = u_j + \frac{h}{2}\left(f(t_j, u_j) + f(t_{j+1}, u_j + hf(t_j, u_j))\right)$$

It can be rewritten as

$$k_1 = f(t_j, u_j)$$

$$k_2 = f(t_j + h, u_j + hk_1)$$

$$u_{j+1} = u_j + h\left(\frac{k_1 + k_2}{2}\right)$$

CLASSICAL RUNGE-KUTTA METHOD

$$k_1 = f(t_j, u_j) \qquad k_2 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2}k_1\right) \qquad k_3 = f\left(t_j + \frac{h}{2}, u_j + \frac{h}{2}k_2\right) \qquad k_4 = f(t_j + h, u_j + hk_3)$$

$$u_{j+1} = u_j + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Why Classical Runge-Kutta method of order 4 is popular?

Minimum number of function evaluation (MNFE) versus order

ORDER	2	3	4	5	6	7	8	⋯
MNFE	**2**	**3**	**4**	**6**	**7**	**9**	**11**	⋯

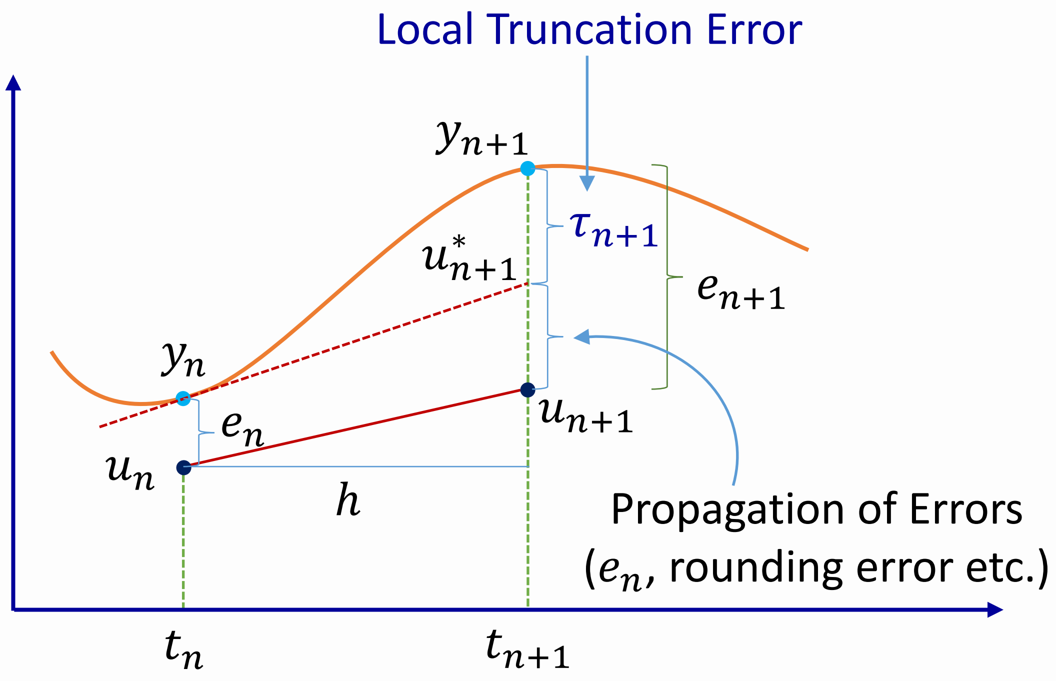
Remark: The order of an s-stage explicit method (RK) cannot be greater than s.

Also, there does not exist a s-stage method (explicit RK) with order s if $s \geq 5$.

A method is said to be convergent if $|u_n - y_n| \to 0, \ \forall n \ \text{as} \ h \to 0$.

Moreover, it is said to be convergence of order $p\ (\geq 1)$ if there exists $C > 0$ such that

$$|u_n - y_n| \leq Ch^p, \qquad \forall n \ \text{as} \ h \to 0 \qquad \Leftrightarrow |u_n - y_n| = \mathcal{O}(h^p)$$



Convergence is the study of the Global error (e_n) and consistency is the study of the local error (τ_n).

L.T.E.: Single Step Numerical Method

$$u_{n+1} = u_n + h \times \phi(t_n, u_n, f(t_n, u_n), h)$$

$$\tau_{n+1} = y_{n+1} - u^*_{n+1}$$

$$\tau_{n+1} = y_{n+1} - y_n - h\phi(t_n, y_n, f(t_n, y_n), h)$$

Consistency: The method is said to be consistent if

$$\frac{1}{h}\tau_{n+1} \to 0, \forall n \ \text{as} \ h \to 0$$

Moreover, it is said to be convergence of order $p(\geq 1)$ if there exists $C > 0$ such that $\left|\frac{1}{h}\tau_{n+1}\right| \leq Ch^p, \ \forall n \ \text{as} \ h \to 0.$

Euler Method: $u_{n+1} = u_n + hf(t_n, u_n)$

$$\tau_{n+1} = y_{n+1} - y_n - hf(t_n, y_n)$$

$$\tau_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(t_n) - y_n - hf(t_n, y_n)$$

$$\tau_{n+1} = y_n + hf(t_n, y_n) + \frac{h^2}{2}y''(t_n) + \cdots - y_n - hf(t_n, y_n)$$

$$\Rightarrow \left|\frac{\tau_{n+1}}{h}\right| = \mathcal{O}(h)$$

The calculation of consistency is essentially based on the application of Taylor series expansion.

Backward Euler Method

$$\left|\frac{\tau_{n+1}}{h}\right| = \mathcal{O}(h)$$

Explicit Midpoint Method

Implicit Midpoint Method

Implicit Trapezoidal Method

Euler-Cauchy Method

$$\left|\frac{\tau_{n+1}}{h}\right| = \mathcal{O}(h^2)$$

$$u_{n+1} = u_n + hf(t_n, u_n); \quad u_0 = y_0$$

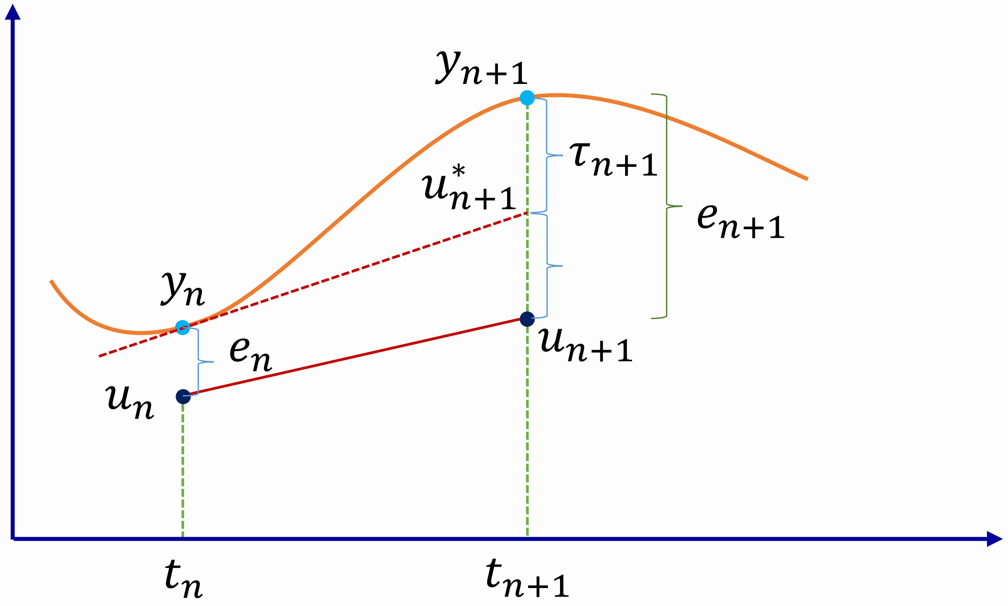
$$\Rightarrow e_{n+1} = y_{n+1} - u_{n+1};\ n = 0, 1, 2, \ldots$$

$$\Rightarrow e_{n+1} = u_{n+1}^* - u_{n+1} + \tau_{n+1}$$

$$\Rightarrow e_{n+1} = y_n + hf(t_n, y_n) - u_n - hf(t_n, u_n) + \tau_{n+1}$$

$$\Rightarrow e_{n+1} = e_n + h\big(f(t_n, y_n) - f(t_n, u_n)\big) + \tau_{n+1}$$

$$\Rightarrow |e_{n+1}| \le |e_n| + h|f(t_n, y_n) - f(t_n, u_n)| + |\tau_{n+1}|$$



Using Lipchitz condition on f, i.e., $|f(t_n, y_n) - f(t_n, u_n)| \le L|y_n - u_n|$ and $|\tau_{n+1}| \le Ch^2$

$$\Rightarrow |e_{n+1}| \le Ch^2 + (1 + Lh)|e_n|$$

By recursion

$$\Rightarrow |e_{n+1}| \le Ch^2 + (1 + Lh)(Ch^2 + (1 + Lh)|e_{n-1}|) = Ch^2 + Ch^2(1 + Lh) + (1 + Lh)^2|e_{n-1}|$$

$$|e_{n+1}| \leq Ch^2 + Ch^2(1+Lh) + (1+Lh)^2|e_{n-1}|$$

$$\Rightarrow |e_{n+1}| \leq Ch^2 + Ch^2(1+Lh) + Ch^2(1+Lh)^2 + (1+Lh)^3|e_{n-2}|$$

$$\Rightarrow |e_{n+1}| \leq Ch^2 + Ch^2(1+Lh) + Ch^2(1+Lh)^2 + \cdots + (1+Lh)^n|e_1|$$

$$\Rightarrow |e_{n+1}| \leq Ch^2 + Ch^2(1+Lh) + Ch^2(1+Lh)^2 + \cdots + Ch^2(1+Lh)^n$$

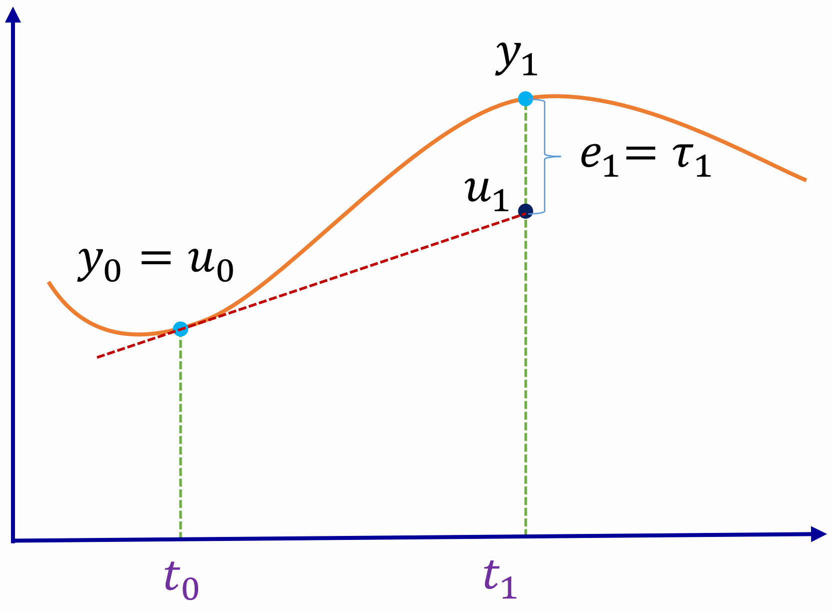
$$\Rightarrow |e_{n+1}| \leq \frac{(1+hL)^{n+1}-1}{(1+hL-1)}Ch^2$$

$$\Rightarrow |e_{n+1}| \leq \frac{(1+hL)^{n+1}-1}{L}Ch$$

Using $1 + hL \leq e^{hL}$

$$|e_{n+1}| \leq \frac{e^{(n+1)hL}-1}{L}Ch$$

$$\Rightarrow |e_{n+1}| \leq \frac{e^{(t_{n+1}-t_0)L}-1}{L}Ch; \quad \forall n$$



Note that: $t_1 = t_0 + h$

$t_2 = t_0 + 2h$

$t_{n+1} = t_0 + (n+1)h$

The total error tends to zero with the same order as consistency error

$$\left|\frac{\tau_{n+1}}{h}\right| = \mathcal{O}(h)$$

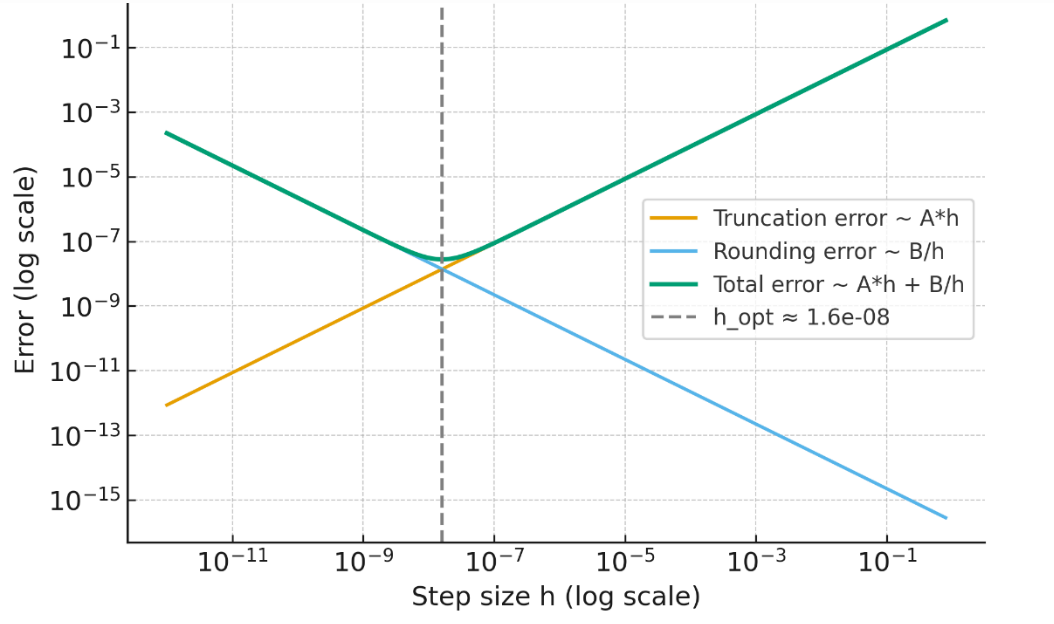
$$\bar{u}_{n+1} = \bar{u}_n + hf(t_n, \bar{u}_n) + \xi_n; \quad \bar{u}_0 = y_0 + \xi_0$$

Presence of rounding error does not allow to conclude that $e \to 0$ as $h \to 0$.

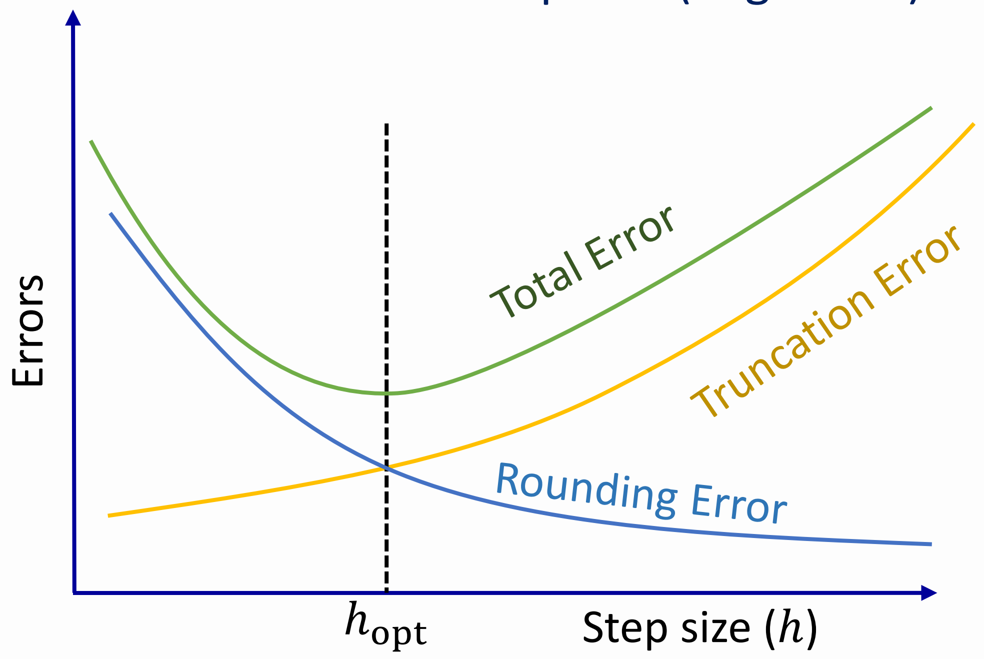
$$\Rightarrow |y_{n+1} - \bar{u}_{n+1}| \le e^{(t_{n+1}-t_0)L}\left(|\xi_0| + \frac{1}{L}\left(Ch + \frac{\xi}{h}\right)\right); \quad \xi = \max_{1\le i\le n+1} \xi_i$$

$$\Rightarrow e_{n+1} \le Ah + \frac{B}{h}$$

ERROR vs. Step Size (Euler)



ERROR vs. Step Size (In general)



Direct convergence proof is difficult specifically for more complicated methods.

Alternatively we can study *convergence* through *consistency* and *stability*.

Stability of the IVP:

$$y' = f(t, y); \quad y(t_0) = y_0, \quad t \in I \qquad (1)$$

Perturbed IVP: $$z' = f(t, z) + \delta(t); \quad z(t_0) = y_0 + \delta_0, \quad t \in I \qquad (2)$$

The IVP (1) is said to be stable on I if for any perturbation $(\delta_0, \delta(t))$ stasifying

$$|\delta_0| < \epsilon, |\delta(t)| < \epsilon, \qquad \forall t \in I$$

There exists $C > 0$ such that

$$|y(t) - z(t)| < C\epsilon$$

The constant C depends in general on the problem data t_0, y_0 and f but not on ϵ.

Zero Stability of One-Step Methods (Analogous to the Stability of IVP):

$$\frac{dy}{dt} = f(t,y); \quad y(t_0) = y_0, t \in [t_0, t_0 + T]$$

Single Step Method: $u_{n+1} = u_n + h\phi(t_n, u_n, f(t_n, u_n); h), \; u_0 = y_0 \qquad (1)$

Single Step Method (perturbed): $z_{n+1} = z_n + h[\phi(t_n, z_n, f(t_n, z_n); h) + \delta_{n+1}], \; z_0 = y_0 + \delta_0$

The numerical method (1) is said to be zero stable if $\exists \; h_o > 0$ and $C > 0$ such that $\forall h \in (0, h_0], \; \forall \epsilon > 0,$

if $|\delta_n| < \epsilon$, for all $0 \leq n \leq N_h$, then $|z_n - u_n| < C\epsilon, \; 0 \leq n \leq N_h$

Both constant C and h_0 may depend on problem's data set $t_0, T, y_o,$ and f but not on ϵ.

This notion of the stability deals with the behaviour of the numerical methods in the limit case $h \to 0$.

Zero Stability of One-Step Method: $u_{n+1} = u_n + h\phi(t_n, u_n, f(t_n, u_n); h),\ u_0 = y_0 \qquad (1)$

$$z_{n+1} = z_n + h[\phi(t_n, z_n, f(t_n, z_n); h) + \delta_{n+1}],\ z_0 = y_0 + \delta_0 \qquad (2)$$

If the increment function ϕ is Lipschitz continuous with respect to the second argument, with content L independent of h and of the nodes $t_n \in [t_0, t_0 + T]$, i.e.,

$$|\phi(t_n, u_n, f(t_n, u_n); h) - \phi(t_n, z_n, f(t_n, z_n); h)| \leq L|u_n - z_n|, \qquad 0 \leq n \leq N_h$$

Then the method (1) is zero stable.

Setting $w_{n+1} = z_{n+1} - u_{n+1}$, from equation (1) and (2) we obtain:

$$w_{n+1} = w_n + h[\phi(t_n, u_n, f(t_n, u_n); h) - \phi(t_n, z_n, f(t_n, z_n); h)] + h\delta_{n+1}$$

$$w_{n+1} = w_n + h[\phi(t_n, u_n, f(t_n, u_n); h) - \phi(t_n, z_n, f(t_n, z_n); h)] + h\delta_{n+1}$$

$$\Rightarrow w_n = w_{n-1} + h[\phi(t_{n-1}, u_{n-1}, f(t_{n-1}, u_{n-1}); h) - \phi(t_{n-1}, z_{n-1}, f(t_{n-1}, z_{n-1}); h)] + h\delta_n$$

Substituting w_{n-1} with the above relation again recursively we obtain:

$$w_n = w_0 + h\sum_{j=1}^{n} \delta_j + h\sum_{j=0}^{n-1} \Big(\phi\big(t_j, u_j, f(t_j, u_j); h\big) - \phi\big(t_j, z_j, f(t_j, z_j); h\big)\Big) \qquad (w_0 = z_0 - u_0 = \delta_0)$$

Using Lipschitz continuity of ϕ

$$|w_n| \le |\delta_0| + h\sum_{j=0}^{n-1} |\delta_{j+1}| + hL\sum_{j=0}^{n-1} |w_j|, \qquad 1 \le n \le N_h$$

$$|w_n| \le |\delta_0| + h\sum_{j=0}^{n-1}|\delta_{j+1}| + hL\sum_{j=0}^{n-1}|w_j|$$

Using Gronwall's Lemma and $|\delta_n| < \epsilon$, we get

$$|w_n| \le \left(|\delta_0| + h\sum_{j=0}^{n-1}|\delta_{j+1}|\right)\exp(nhL)$$

$$|w_n| \le (1+\underbrace{nh}_{\le T})\epsilon\exp(\underbrace{nh}_{\le T}L) \le \underbrace{(1+T)\exp(TL)\,\epsilon}_{=C}$$

Hence the method (1) is zero stable

Gronwall's Lemma (Discrete) Let k_n be a nonnegative sequence and ϕ_n a sequence such that

$$w_0 \le g_0 \quad \text{and} \quad w_n \le (g_0 + c_n) + \sum_{j=0}^{n-1} k_s w_s\,, \qquad n \ge 1$$

If $g_0 \ge 0$, $c_n \ge 0$, then $\quad w_n \le (g_0 + c_n)\exp\left(\sum_{j=0}^{n-1} k_s\right), \qquad n \ge 1$

Convergence Theorem (Lax-Richtmyer Theorem)

If the numerical method is stable then

$$|y_n - u_n| \leq \left(|y_0 - u_0| + T\left(\frac{\tau}{h}\right)\right)e^{TL}, \qquad 1 \leq n \leq N_h, \qquad \tau = \max_{0\leq n\leq N_{h-1}} |\tau_{n+1}(h)|$$

If $|y_0 - u_0| \leq C_1 h^p \ (= \mathcal{O}(h^p))$ and $\frac{\tau}{h} = \mathcal{O}(h^p)$ (consistency order p) then $|y_n - u_n| = \mathcal{O}(h^p)$

A numerical method is convergent of order *p* if it is consistent of order *p* and zero-stable.

Consistency + Stability $\Rightarrow$ Convergence

Sketch of the proof:

$$u_{n+1} = u_n + h\phi(t_n, u_n, f(t_n, u_n); h), \ \ u_0 = y_0 \qquad (1)$$

$$y_{n+1} = y_n + h\phi(t_n, y_n, f(t_n, y_n); h) + \tau_{n+1} \qquad (2)$$

Proceed as before for the case of zero stability.

Consider the following problem

$$\frac{dy}{dt} = \lambda y; \quad y(0) = 1 \quad \text{Exact Solution: } y(t) = e^{\lambda t}$$

Apply the Euler Method

$$u_{n+1} = u_n + \lambda h u_n \qquad \Rightarrow u_{n+1} = (1 + \lambda h) u_n$$

$$\Rightarrow u_n = (1 + \lambda h)^n u_0 \qquad \text{given } u_0 = 1$$

$$\Rightarrow u_n = \exp(\ln(1 + \lambda h)^n) \qquad \left(h = \frac{t_n}{n}\right)$$

Let $n \to \infty, h \to 0$ such that $nh = t_n$

$$\Rightarrow u_n = \lim_{n\to\infty} \exp\left(t_n \lambda \frac{\ln\left(1 + \frac{t_n \lambda}{n}\right)}{\frac{t_n \lambda}{n}} \right) \Rightarrow u_n = \exp(\lambda t_n)$$

But in practice $h \nrightarrow 0$

Lets take $\lambda = -10$ & $h = 0.4$

$$\Rightarrow u_n = (-3)^n$$

The solution blow-up and oscillates

We observe that the numerical solution will decay if

$$|(1 + \lambda h)| < 1 \Rightarrow -2 < \lambda h < 0$$

Remark. *Zero-stability, while necessary for convergence, is not sufficient for practical purposes. In real computations, one often requires stronger notions of stability (such as absolute stability, A-stability, or L-stability) to ensure reliable performance of the numerical method*

Model Equation: Let us consider the following IVP:

$$\frac{dy}{dt} = f(t, y); \quad y(t_0) = y_0, t \in [t_0, b]$$

The behavior of solution of above IVP in the neighborhood of any point $(\bar{t}, \bar{y})$ can be predicted by the linearized form of DE.

The function $f(t, y)$can be linearized in the neighborhood of the point $(\bar{t}, \bar{y})$ by expending it into the Taylor series as

$$f(t, y) = f(\bar{t}, \bar{y}) + (t - \bar{t})\, f_t(\bar{t}, \bar{y}) + (y - \bar{y}) f_y(\bar{t}, \bar{y}) + \text{higher order terms}$$

Define: $\mu = f_t(\bar{t}, \bar{y})$, $\quad \lambda = f_y(\bar{t}, \bar{y})$, $\quad c = f(\bar{t}, \bar{y}) - \bar{y}\lambda + (t - \bar{t})\mu$

The given ode can be written as $\quad y' \approx \lambda y + c$

Substituting $\omega = y + \left(\frac{c}{\lambda}\right) + \left(\frac{\mu}{\lambda^2}\right)$

$$y' - \frac{\mu}{\lambda} \approx \lambda\left[\omega - \left(\frac{c}{\lambda}\right) - \left(\frac{\mu}{\lambda^2}\right)\right] + c \;\Rightarrow\; \omega' = \lambda\omega \quad \text{(model equation)} \qquad \text{Exact solution: } \omega = ke^{\lambda t}$$

For practical purposes (similar to lab scale test), it is sufficient to check stability for the linear equation $\omega' = \lambda\omega$.

Absolute stability is typically analysed using the test equation $y' = \lambda y$, since it captures several essential features of more general differential equations.

For broader applicability, λ is taken to be complex, $\lambda = \lambda_R + i\lambda_I$, with the restriction $\lambda_R \leq 0$ to ensure that the exact solution remains non-growing (i.e., stable) in the continuous case.

Absolute Stability: A numerical method for approximating

$$y'(t) = \lambda y; \;\; t > 0; \;\; \mathrm{Re}(\lambda) < 0\,, \;\; y(0) = 1$$

is absolutely stable if $|u_n| \to 0$ as $t_n \to \infty$

Region of absolute stability = $\{\lambda h \in \mathbb{C}: \text{the method is absolutely stable at } \lambda h\}$

Abolute Stability $\implies$ Zero Stability

Please note that a single step method when applied to the test equation $y'(t) = \lambda y$ leads to

$$u_{n+1} = E(\lambda h)u_n; \;\; n = 0, 1, 2, \ldots$$

We call single step method absolutely stable if $|E(\lambda h)| < 1$

Absolute Stability of Euler Method:

Applying the Euler method on the test equation:

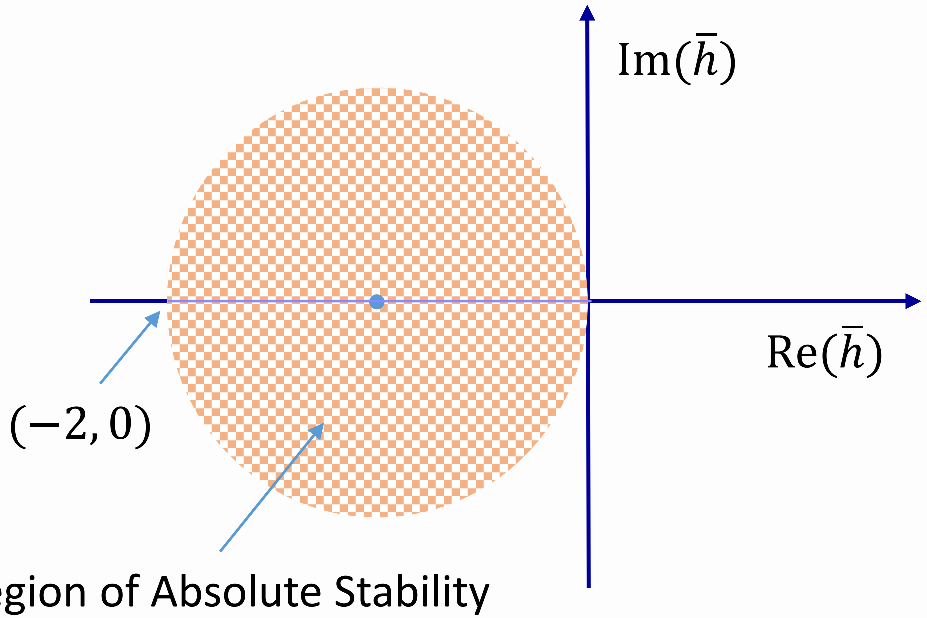
$$u_{n+1} = u_n + h\,\lambda u_n \;\; \Rightarrow u_{n+1} = (1 + \underbrace{\lambda h}_{\bar{h}})u_n$$

$$\bar{h} = x + iy$$

$$|1 + \lambda h| = |1 + x + iy| = \sqrt{(1+x)^2 + y^2}$$

For Absolute stability, we require

$$\sqrt{(1+x)^2 + y^2} < 1 \Rightarrow (1+x)^2 + y^2 < 1$$



Region of Absolute Stability: $\{\bar{h} \in \mathbb{C} : |\bar{h} + 1| < 1\}$

Absolute Stability of Backward Euler Method: $u_{n+1} = u_n + h\, f_{n+1}$

$$\Rightarrow u_{n+1} = u_n + h\,\lambda u_{n+1} \quad \Rightarrow u_{n+1} = \frac{1}{(1-\lambda h)} u_n$$

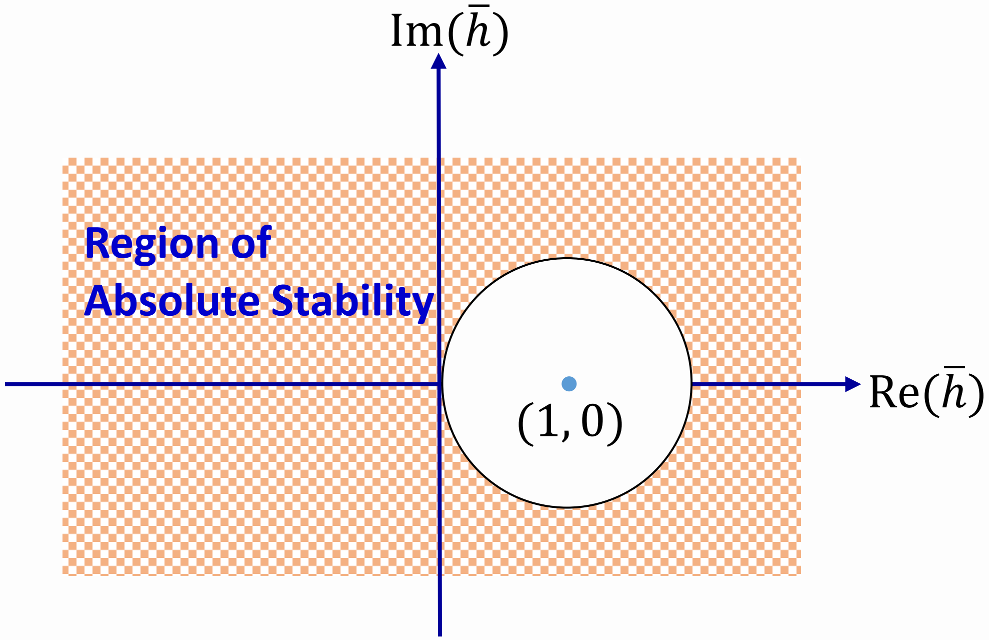
$$\Rightarrow E(\lambda h) = \frac{1}{(1-\lambda h)}$$

For Absolute stability, we require

$$\left|\frac{1}{(1-\lambda h)}\right| < 1 \;\Rightarrow |1-\lambda h| > 1 \qquad (\lambda h = \bar{h} = x + iy)$$

$$\Rightarrow |(1-x) - iy| > 1 \quad \Rightarrow (1-x)^2 + y^2 > 1$$

Region of Absolute Stability: $\{\bar{h} \in \mathbb{C} : |\,\bar{h} - 1\,| > 1\}$



The region of absolute stability is the entire complex plan except a disk centred at $(1, 0)$ with radius 1.

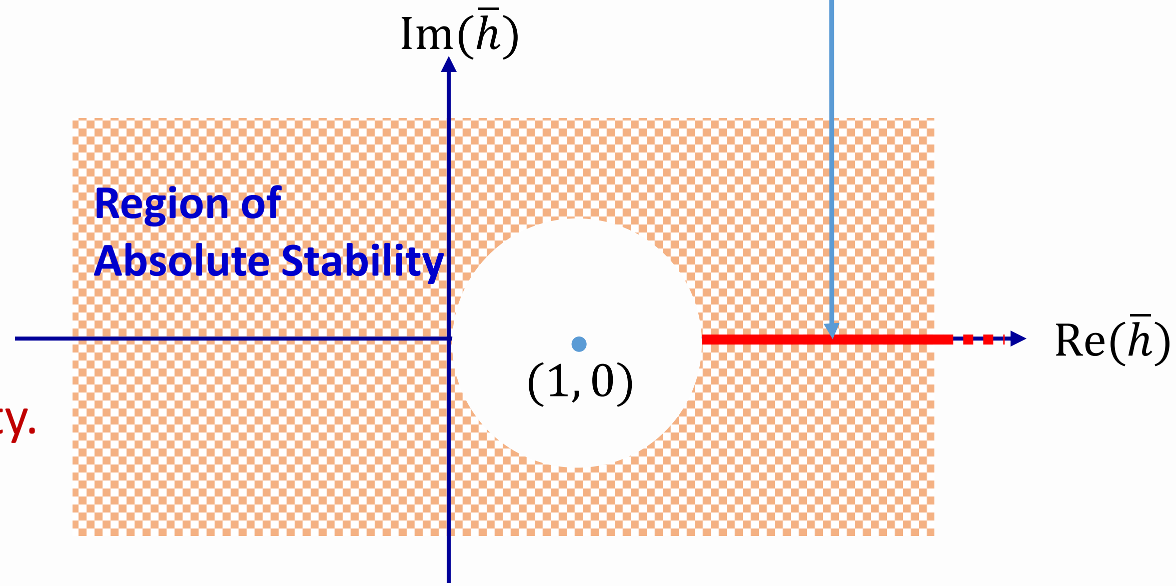
Backward Euler Method is unconditionally absolutely stable.

A method is called **A-stable** if its region of absolute stability includes the entire left half of the complex plane.

Suppose we solve $y'(t) = 5y(t),\ y(0) = 1$ using backward Euler method

If we pick any $5h > 2 \ \Rightarrow h > 2/5$

(Absolute stability region)



Since λh is from the region of stability, the solution will decay and tends to 0 as t tends to ∞.

However, the exact solution of the problem tends to infinity.

Why this is not a contradiction of A-stability?

Backward Euler is **A-stable** because for $\mathrm{Re}(\lambda) < 0$ (decaying continuous problems), its numerical solution will also decay (and not blow up). A-stability says nothing about problems with $\mathrm{Re}(\lambda) > 0$ (growing problems): the method can (and here does) artificially damp them if the step is too large.

Absolute Stability of Trapezoidal Method: $u_{n+1} = u_n + \frac{h}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right)$

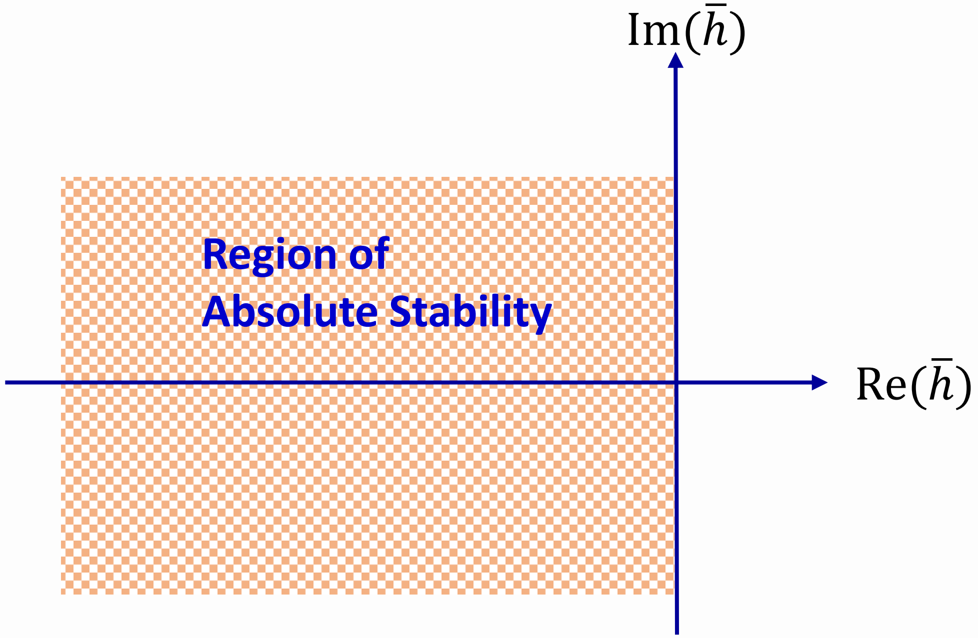
$$\Rightarrow u_{n+1} = u_n + \frac{h}{2}(\lambda u_n + \lambda u_{n+1}) \Rightarrow \left(1 - \frac{\bar{h}}{2}\right) u_{n+1} = \left(1 + \frac{\bar{h}}{2}\right) u_n \quad \Rightarrow u_{n+1} = \frac{\left(1 + \frac{\bar{h}}{2}\right)}{\left(1 - \frac{\bar{h}}{2}\right)} u_n$$

$$\Rightarrow E(\bar{h}) = \frac{\left(1 + \frac{\bar{h}}{2}\right)}{\left(1 - \frac{\bar{h}}{2}\right)}$$

For Absolute stability, we require $\left(1 + \frac{\bar{h}}{2}\right) < \left(1 - \frac{\bar{h}}{2}\right)$

$$\Rightarrow (2 + \bar{h}) < (2 - \bar{h}) \Leftrightarrow \mathrm{Re}(\bar{h}) < 0$$

Region of Absolute Stability: $\{\bar{h} \in \mathbb{C} : \mathrm{Re}(\bar{h}) < 0\}$



Implicit Trapezoidal Method is **A stable.**

Absolute Stability of Euler-Cauchy Method (Heun's Method)

$$u_{n+1} = u_n + \frac{h}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_n + hf(t_n, u_n))\right)$$

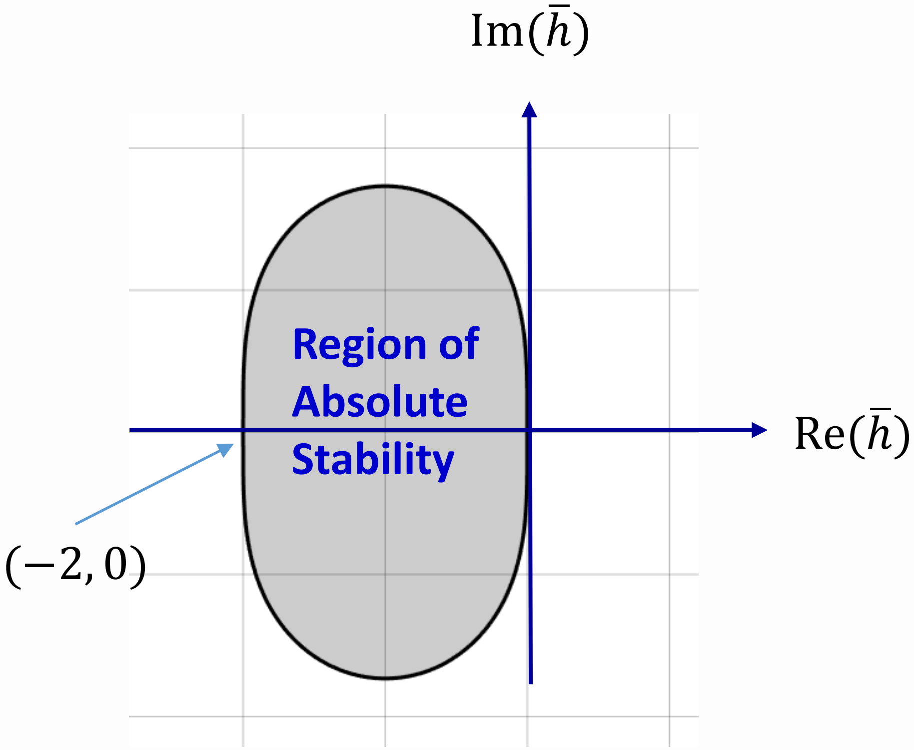
$$\Rightarrow u_{n+1} = u_n + \frac{h}{2}\left(\lambda u_n \; + \lambda u_n\,(1 + \lambda h)\right)$$

$$\Rightarrow u_{n+1} = \left(1 + \frac{\lambda h}{2} + \frac{\lambda h}{2} + \frac{\lambda^2 h^2}{2}\right) u_n$$

$$\Rightarrow u_{n+1} = \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) u_n$$

$$\Rightarrow E(\bar{h}) = \left(1 + \bar{h} + \frac{\bar{h}^2}{2}\right)$$

Region of Absolute Stability: $\left\{\bar{h} \in \mathbb{C}: \left|1 + \bar{h} + \frac{\bar{h}^2}{2}\right| < 1\right\}$



Absolute Stability of Euler-Cauchy Method (Heun's Method)

For real $\lambda < 0$:

$$\left|\left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right)\right| < 1 \quad \Rightarrow -1 < \left(1 + \lambda h + \frac{\lambda^2 h^2}{2}\right) < 1$$

$$\Rightarrow -1 < \frac{1}{2}(2 + 2\lambda h + \lambda^2 h^2) < 1$$

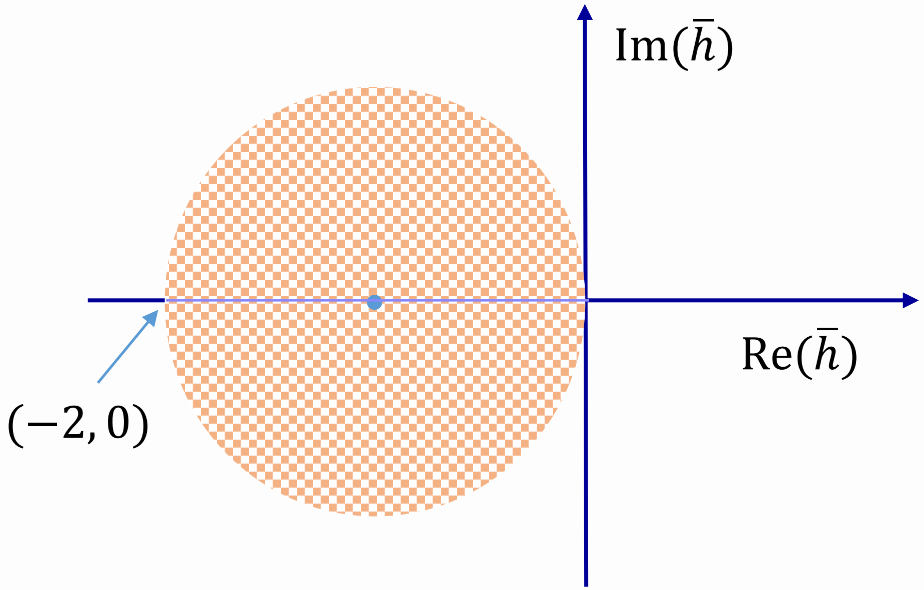
$$\Rightarrow \underbrace{-2 < (1 + (1 + \lambda h)^2)}_{\text{always satisfied}} < 2$$

$$\Rightarrow (1 + (1 + \lambda h)^2) < 2 \quad \Rightarrow (1 + \lambda h)^2 < 1 \quad \Rightarrow -1 < 1 + \lambda h < 1 \quad \Rightarrow -2 < \lambda h < 0$$

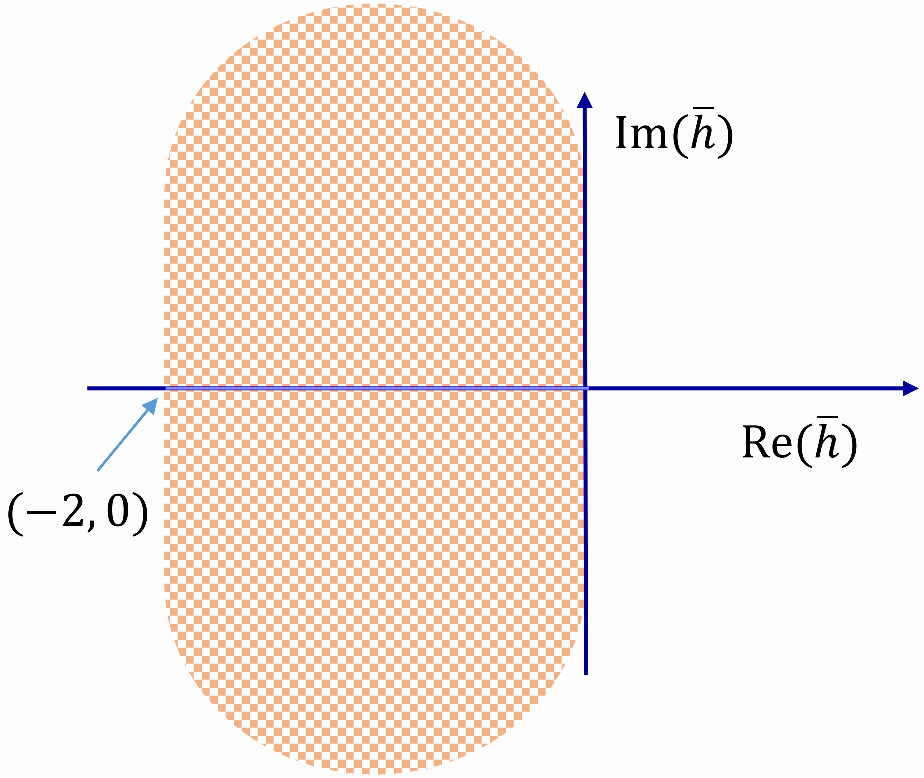
$\Rightarrow \lambda h \in (-2, 0)$ Same as Euler Method

Region of Absolute Stability of Explicit Methods

Euler Method

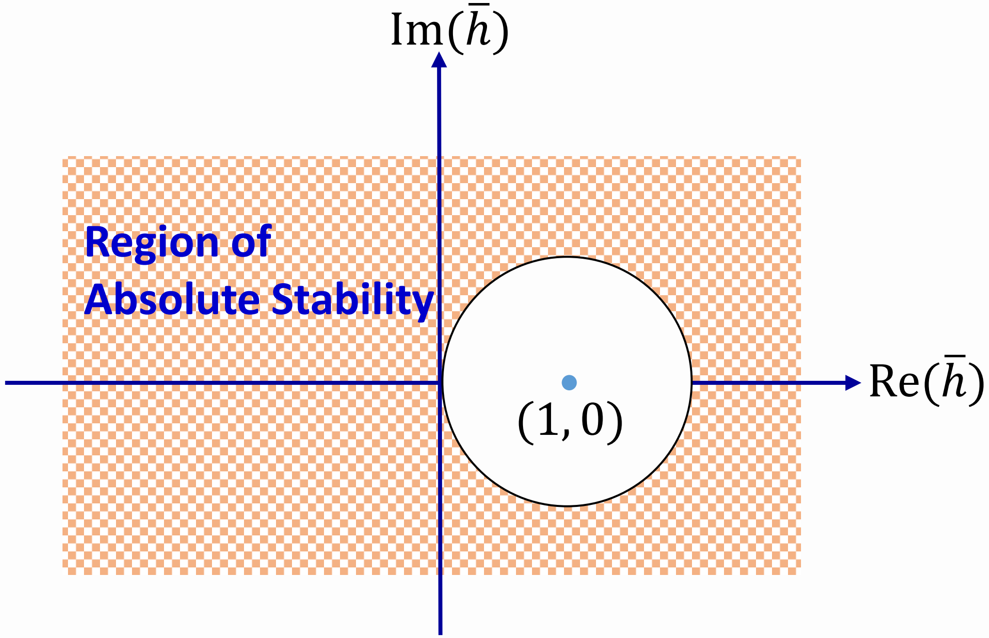


Euler-Cauchy Method

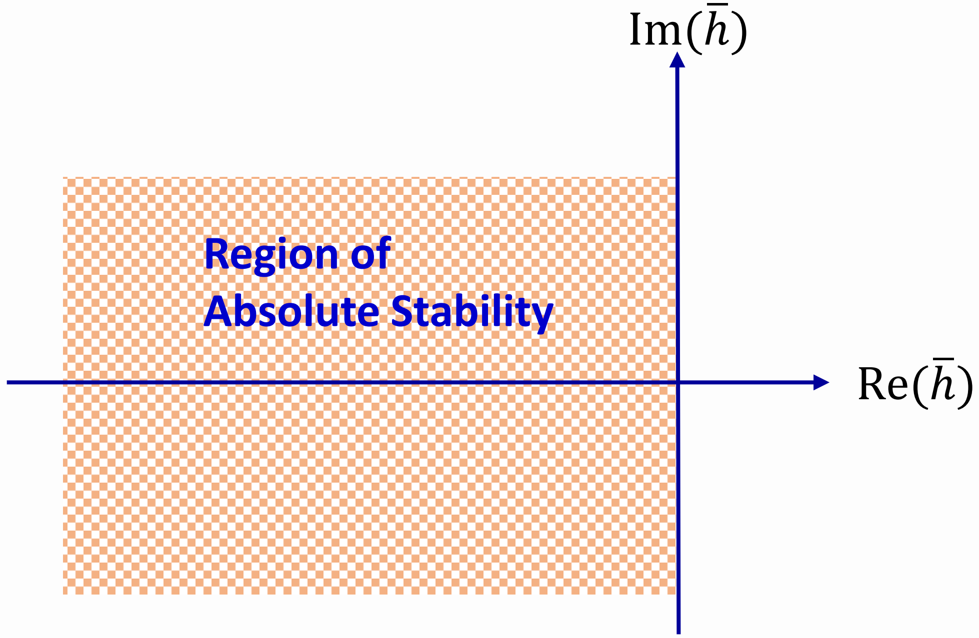


Region of Absolute Stability of Implicit Methods

Backward Euler Method



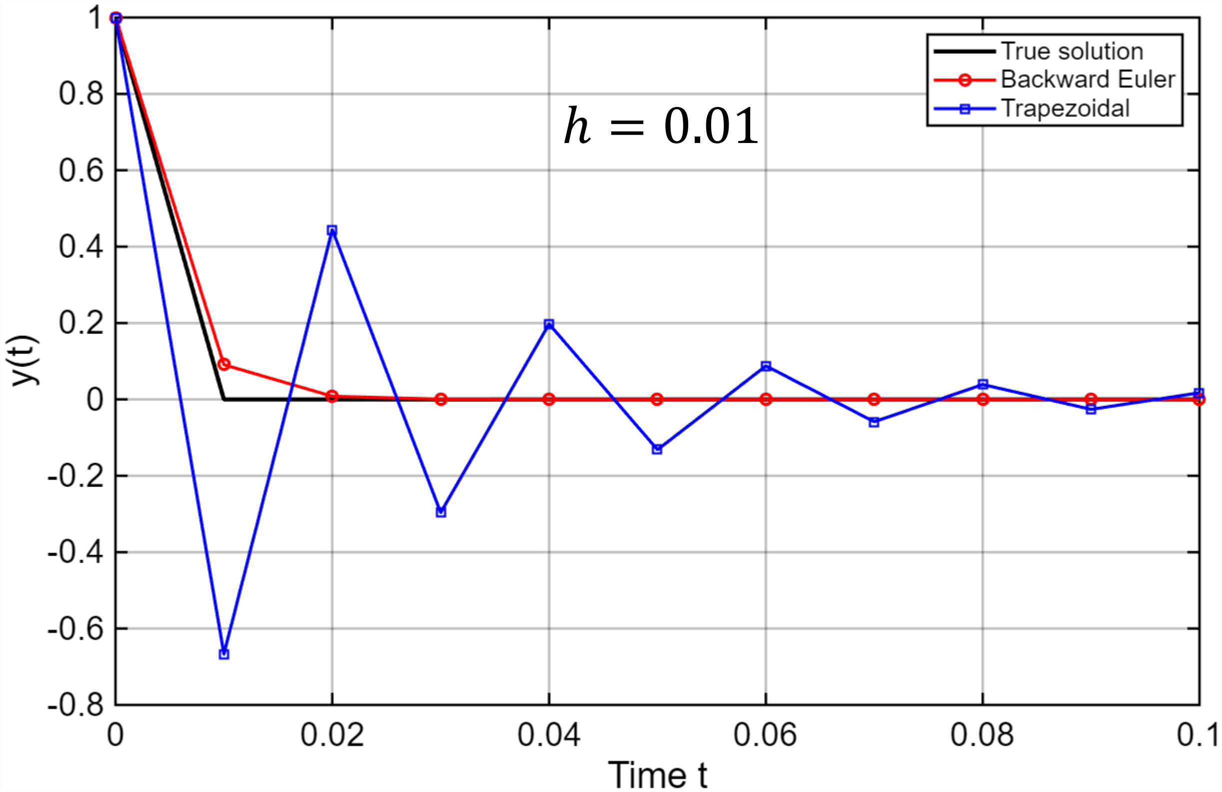
Trapezoidal Method



Consider solution of a stiff IVP: $y'(t) = -1000y;\ \ y(0) = 1$

If we want to solve it using explicit method, e.g., Euler Method then $h < 0.0001$.

Therefore we need to use an implicit method.



Implicit Midpoint Method

$$u_{n+1} = u_n + hf(t_{n+1}, u_{n+1})$$

Implicit Trapezoidal Method

$$u_{n+1} = u_n + \frac{h}{2}\left(f(t_n, u_n) + f(t_{n+1}, u_{n+1})\right)$$

L-Stability

A numerical method for solving ODEs is called **L-stable** if

1. It is **A-stable** (its region of absolute stability includes the entire left half of the complex plane), and

2. Its **stability function** $E(\bar{h})$ satisfies $\lim_{\bar{h}\to-\infty} E(\bar{h}) = 0$

Backward Euler Method:

$$\Rightarrow E(\lambda h) = \frac{1}{(1-\lambda h)} \Rightarrow E(\lambda h) \to 0 \text{ as } \lambda h \to -\infty$$

L-stable

Trapezoidal Method:

$$\Rightarrow E(\lambda h) = \frac{(1+\lambda h/2)}{(1-\lambda h/2)} \Rightarrow E(\lambda h) \to -1 \text{ as } \lambda h \to -\infty$$

Only A-stable and not L-stable

Note: L-stability adds the property that **very stiff components decay to zero rapidly**.

This eliminates spurious oscillations and makes the method especially effective for stiff problems.

CONCLUDING REMARKS

- Implicit one-step methods (examined so far) → *unconditionally absolutely stable*.
- Explicit schemes (examined so far) → *conditionally absolutely stable*.
- This is not a general rule. Some implicit schemes are unstable or only conditionally stable. However, no explicit scheme is *unconditionally absolutely stable*.

Thank You