Basics
Finite Difference Method

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Basic CFD equations

- Conservation of Mass
- Conservation of Momentum
- Conservation of Energy

Reynolds Transport Theorem

- Let **B** be any property of the fluid (mass, momentum, energy)
- Let $b = \frac{dB}{dm}$ be the intensive value of **B** (amount of **B** per unit mass) in any small element of the fluid

$$\left(\frac{d\boldsymbol{B}}{dt}\right)_{MV} = \int_{\boldsymbol{V}} \left[\frac{\boldsymbol{D}}{\boldsymbol{Dt}}(\rho b) + \rho b \nabla . \boldsymbol{v}\right] d\boldsymbol{V}$$

$$\frac{\boldsymbol{Df}}{\boldsymbol{Dt}} = \frac{\boldsymbol{\partial f}}{\boldsymbol{\partial t}} + \boldsymbol{v}.\nabla \boldsymbol{f}$$

General Conservation Equation

Change of $\boldsymbol{\phi}$ over time $\boldsymbol{\Delta}$**t** within the material volume	=	Surface flux of $\boldsymbol{\phi}$ over time $\boldsymbol{\Delta}$**t** across the control volume	+	Source/sink of $\boldsymbol{\phi}$ over time $\boldsymbol{\Delta}$**t** within the control volume
Term I		Term II		Term III

$$\underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\text{unsteady term}} + \underbrace{\nabla.(\rho\boldsymbol{v}\phi)}_{\text{Convection term}} = \underbrace{\nabla.(\Gamma\nabla\phi)}_{\text{Diffusion term}} + \underbrace{Q}_{\text{Source term}}$$

Basic CFD equations

$$\frac{\partial \rho}{\partial t} + \nabla . (\rho \boldsymbol{v}) = 0$$

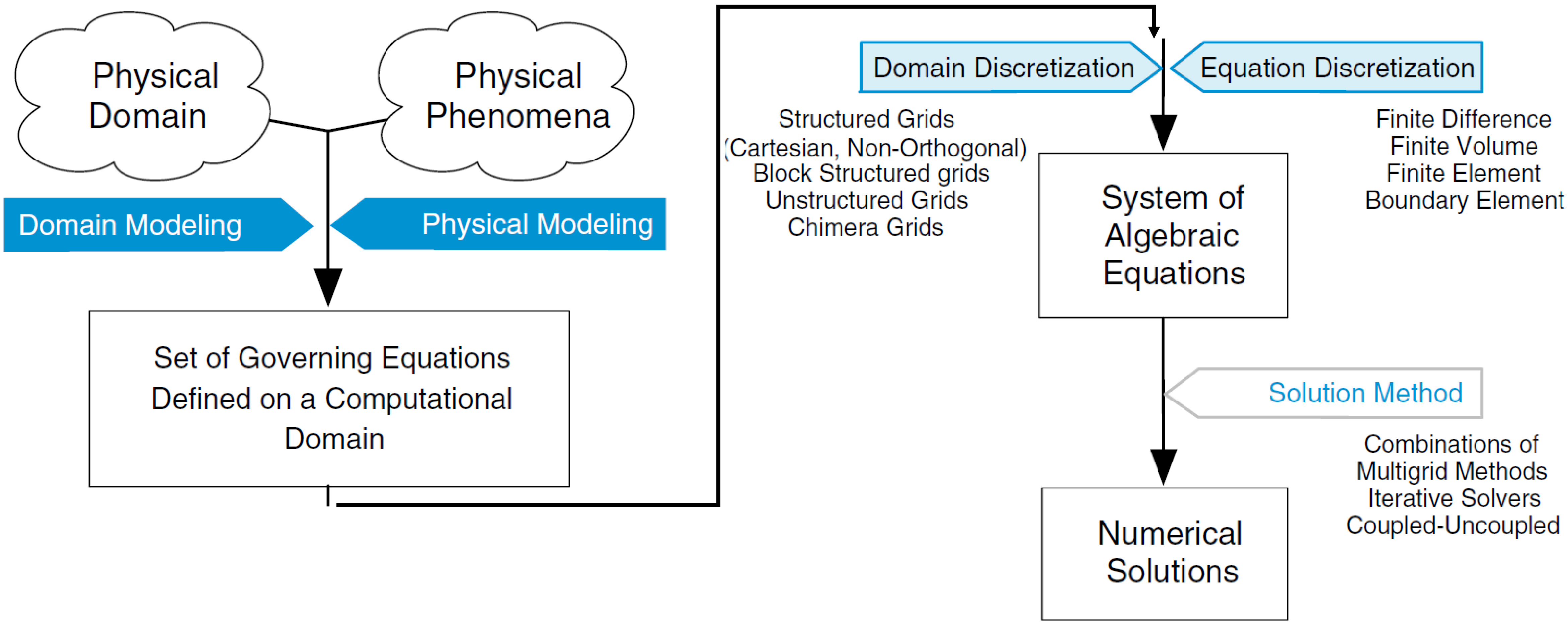
$$\frac{\partial (\rho \boldsymbol{v})}{\partial t} + \nabla . [\rho \boldsymbol{v}\boldsymbol{v}.\boldsymbol{n} - n\boldsymbol{T_s}] - f_b = 0$$

$$\frac{\partial (\rho \boldsymbol{E})}{\partial t} + \nabla . [\rho \boldsymbol{E}\boldsymbol{u} \; - \boldsymbol{v}\boldsymbol{T_s} + \boldsymbol{q}] - f_b . \boldsymbol{v} = 0$$

Navier-Stokes Equation

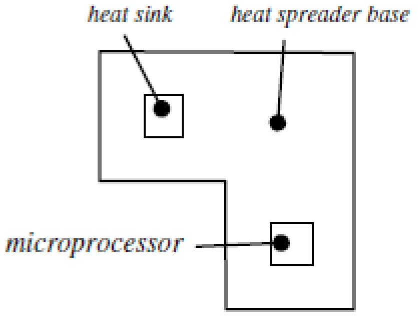
$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla.(\rho \boldsymbol{v} \otimes \boldsymbol{v}) = -\nabla.p\boldsymbol{I} + \nabla.2\mu D + \rho \boldsymbol{g}$$

Discretization

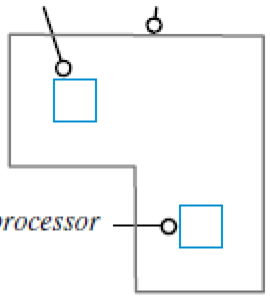


Discretization

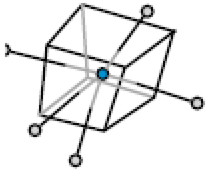
Domain Modeling



Physical Modeling



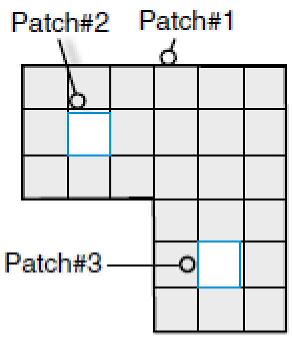
Equations Discretization



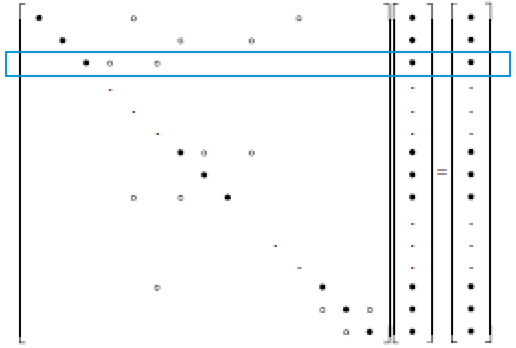
$$\underbrace{\frac{\partial(\rho\phi)}{\partial t}}_{\text{transient term}}+\underbrace{\nabla\cdot(\rho\mathbf{v}\phi)}_{\text{convection term}}=\underbrace{\nabla\cdot(\Gamma\nabla\phi)}_{\text{diffusion term}}+\underbrace{Q}_{\text{source term}}$$

$$a_C\phi_C+\sum_{F\sim NB(C)}a_F\phi_F=b_C$$

Domain Discretization

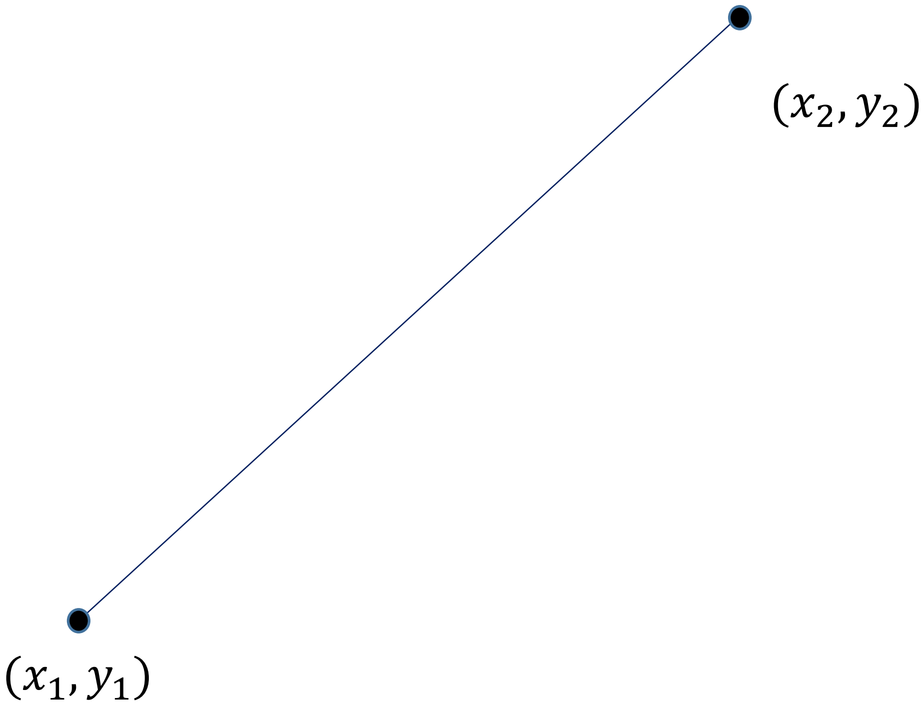


Solution Method

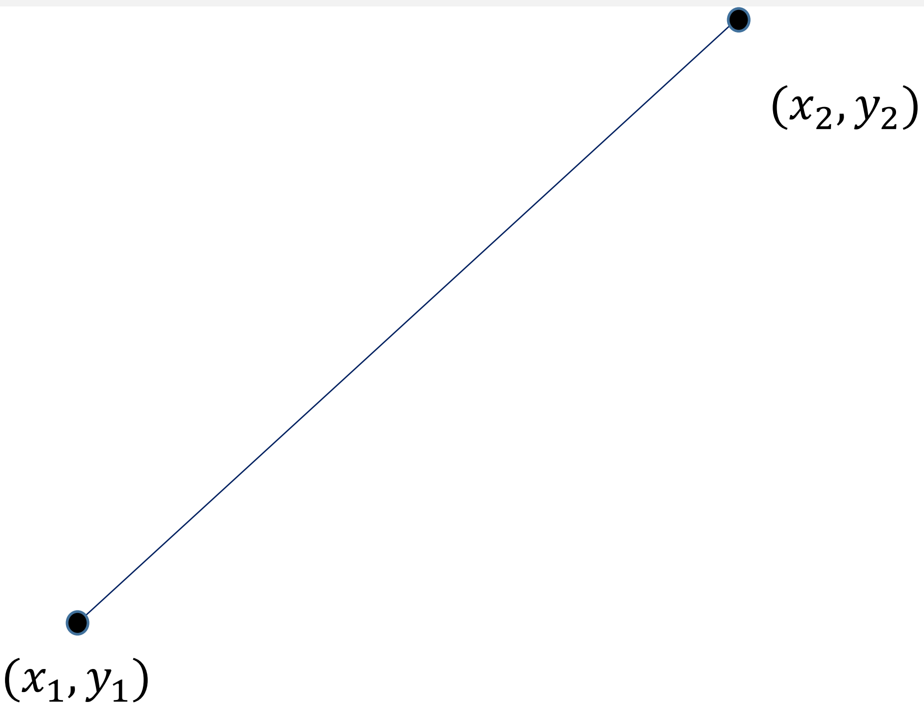


Taylor Series and Numerical Analysis

Straight Line

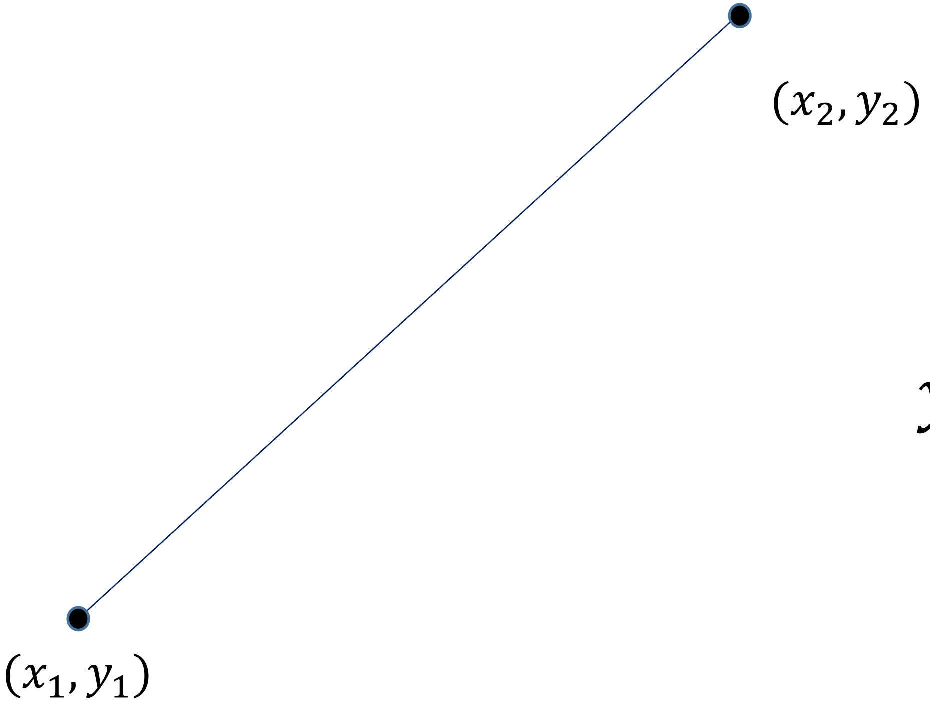


Straight Line



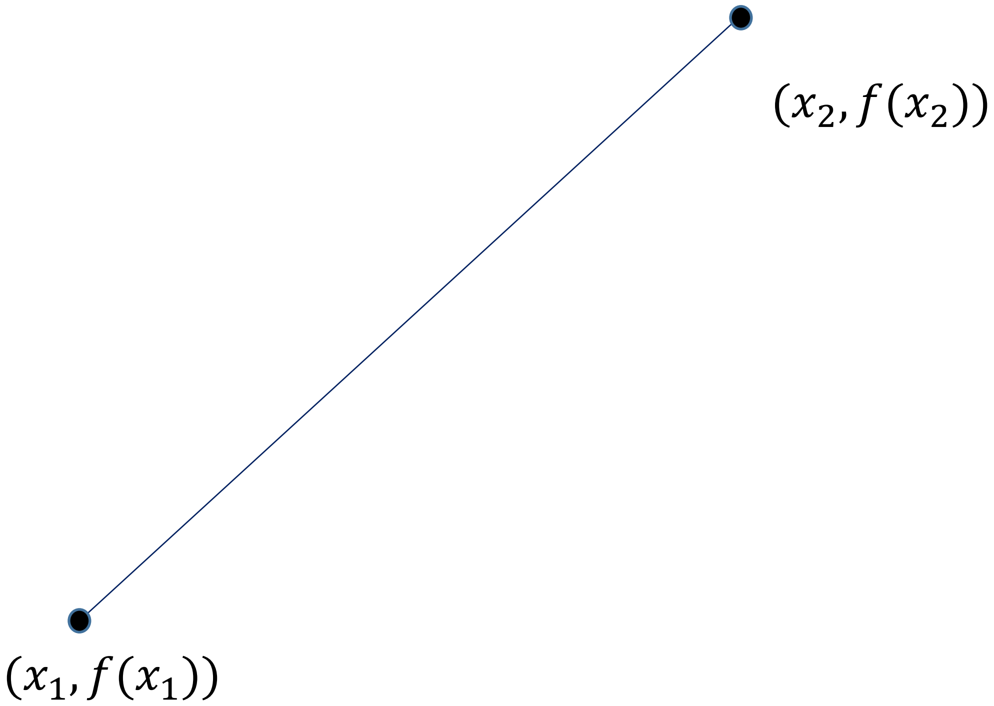
$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

Straight Line



$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

Straight Line



$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

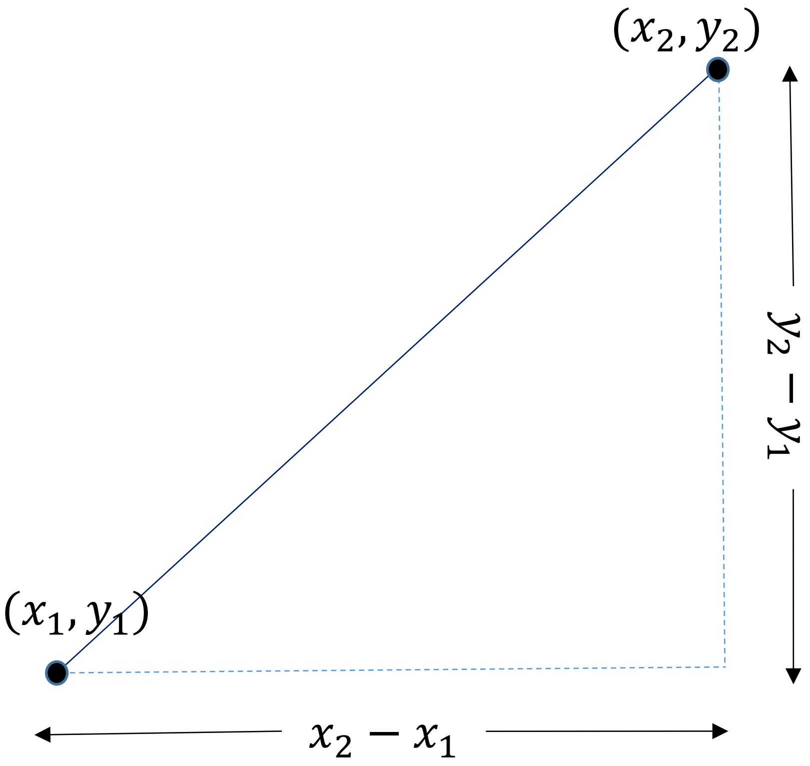
$$y = f(x)$$

$$y_1 = f(x_1)$$

$$y_2 = f(x_2)$$

$$f(x) = f(x_1) + \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)(x - x_1)$$

Slope

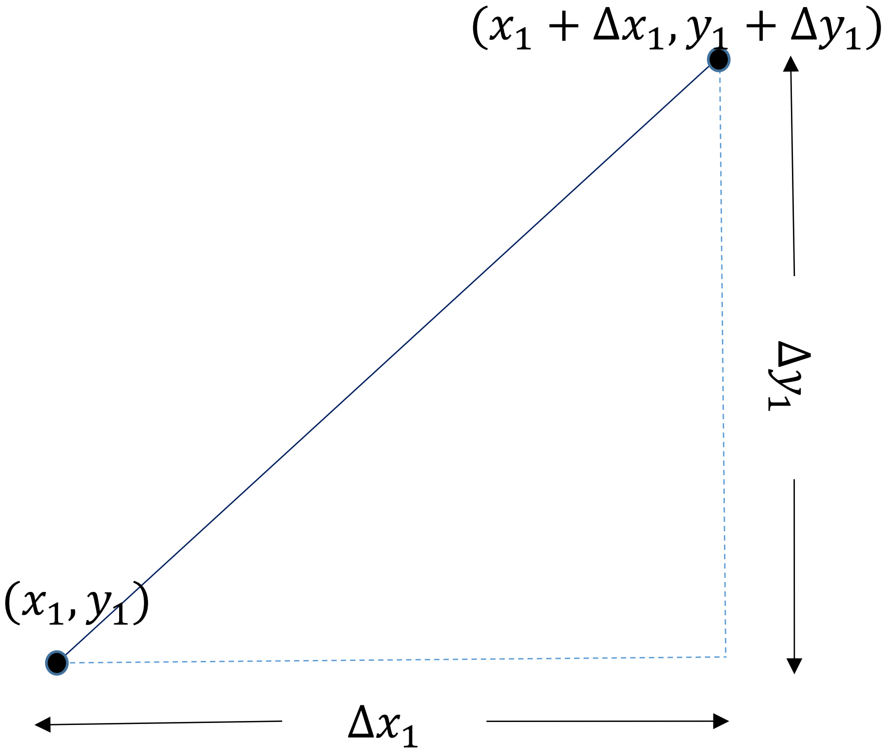


$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$y = y_1 + m(x - x_1)$$

Slope

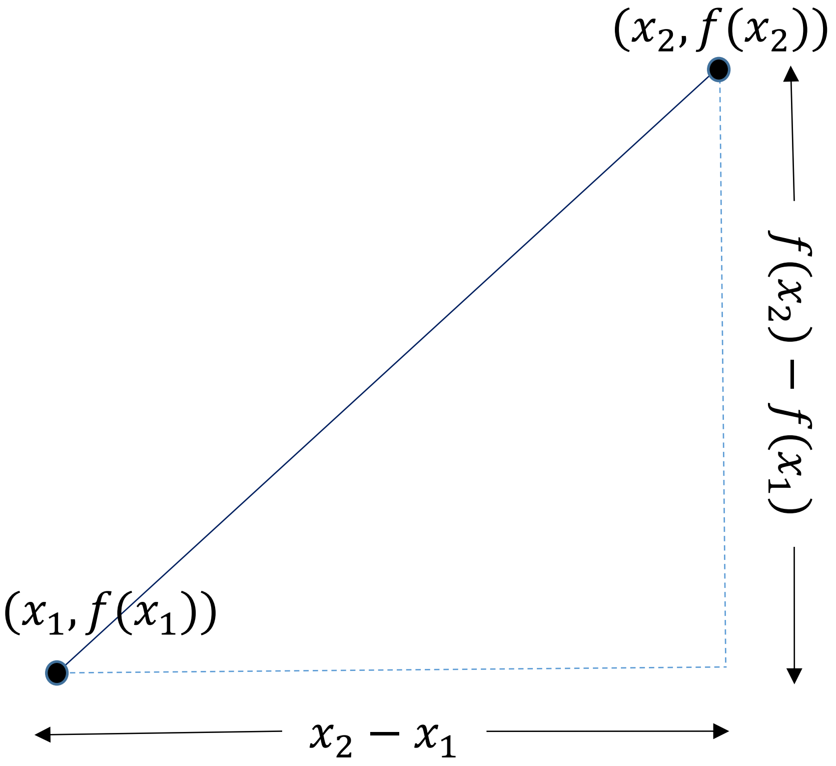


$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$m = \frac{\Delta y_1}{\Delta x_1}$$

$$y = y_1 + m(x - x_1)$$

Slope



$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

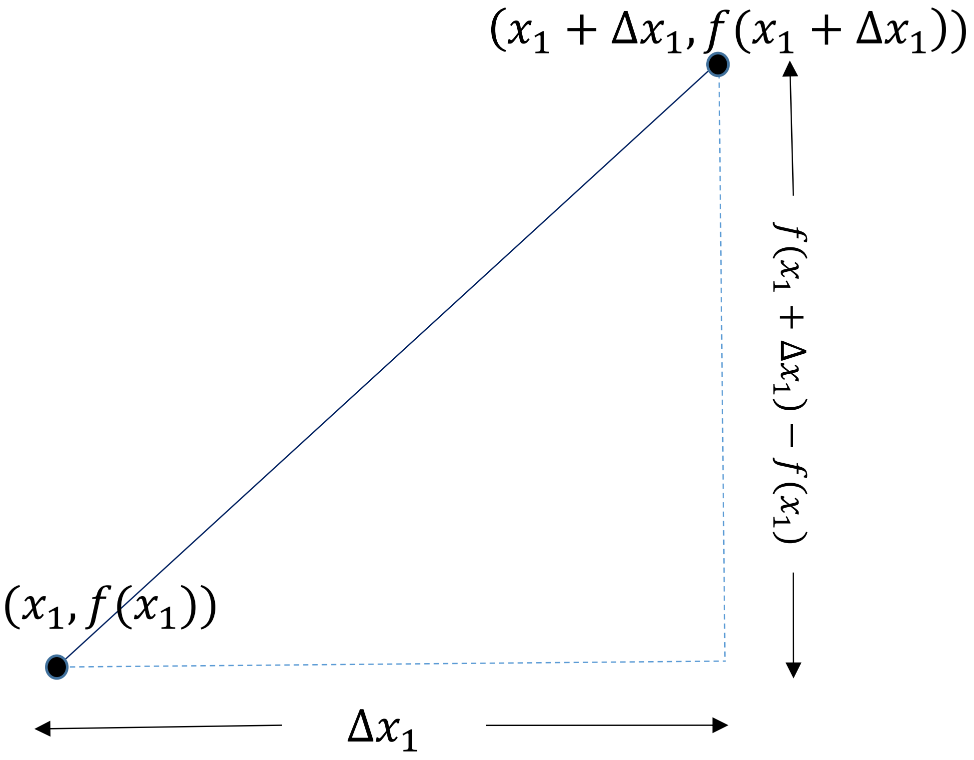
$$y = f(x)$$

$$y_1 = f(x_1)$$

$$y_2 = f(x_2)$$

$$f(x) = f(x_1) + \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1}\right)(x - x_1)$$

Slope



$$y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

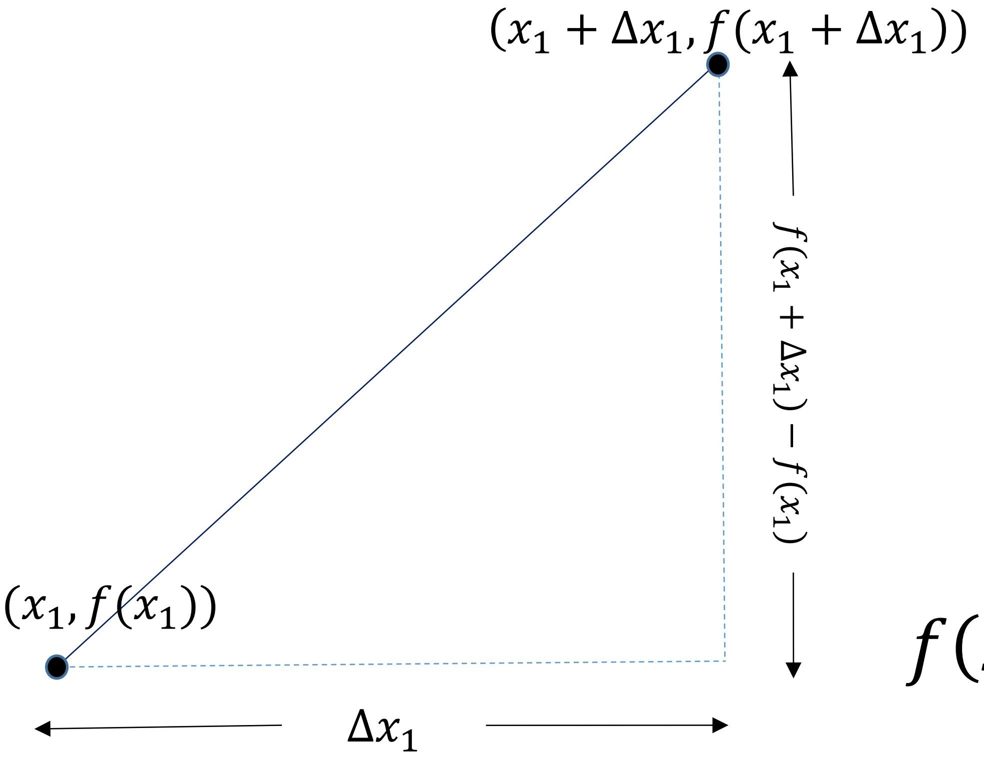
$$y = f(x)$$

$$y_1 = f(x_1)$$

$$y_2 = f(x_1 + \Delta x_1)$$

$$f(x) = f(x_1) + \left(\frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x_1}\right)(x - x_1)$$

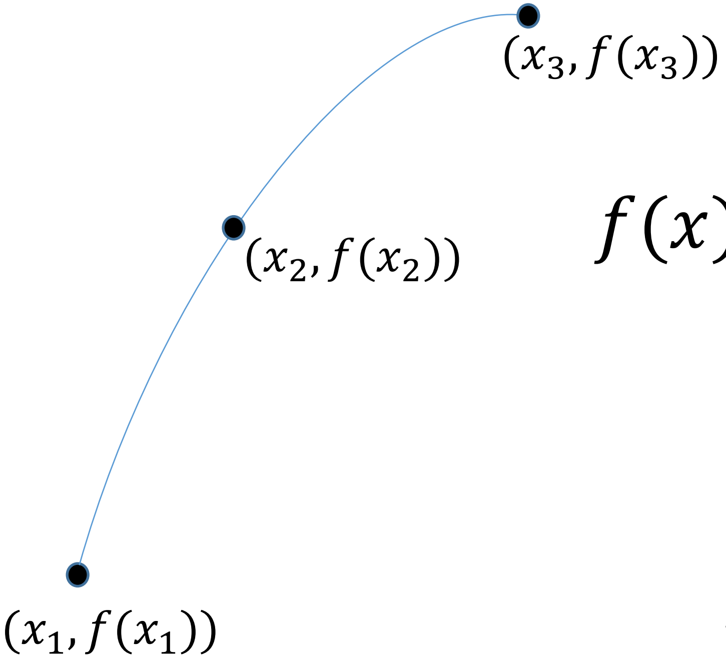
Slope



$$y = f(x)$$

$$f(x) = f(x_1) + \left(\frac{dy}{dx}\right)_{x=x_1} (x - x_1)$$

$$f(x) = f(x_1) + f'(x_1)(x - x_1)$$

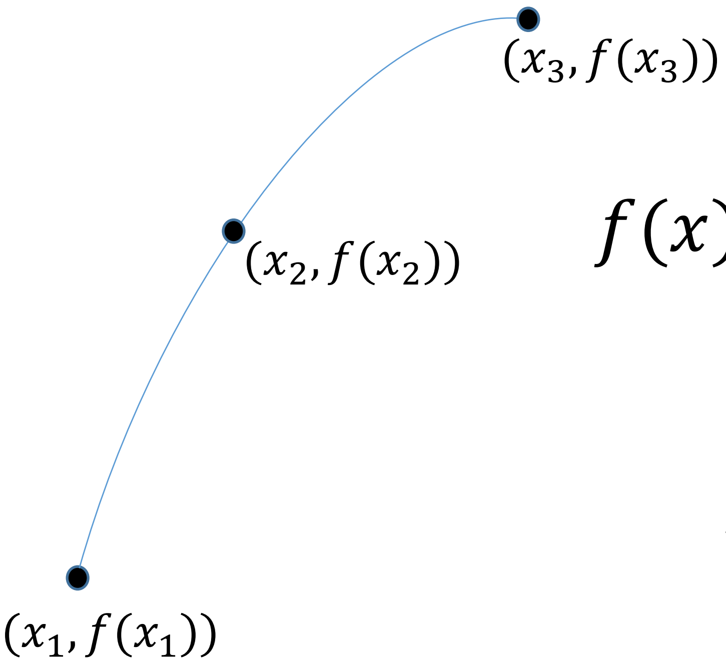


$$f(x) = a(x - x_1)(x - x_2) + b(x - x_1) + c$$

$$f(x_1) = c$$

$$f(x_2) = b(x_2 - x_1) + f(x_1)$$

$$f(x_3) = a(x_3 - x_1)(x_3 - x_2) + b(x_3 - x_1) + f(x_1)$$



$$f(x) = a(x - x_1)(x - x_2) + b(x - x_1) + c$$

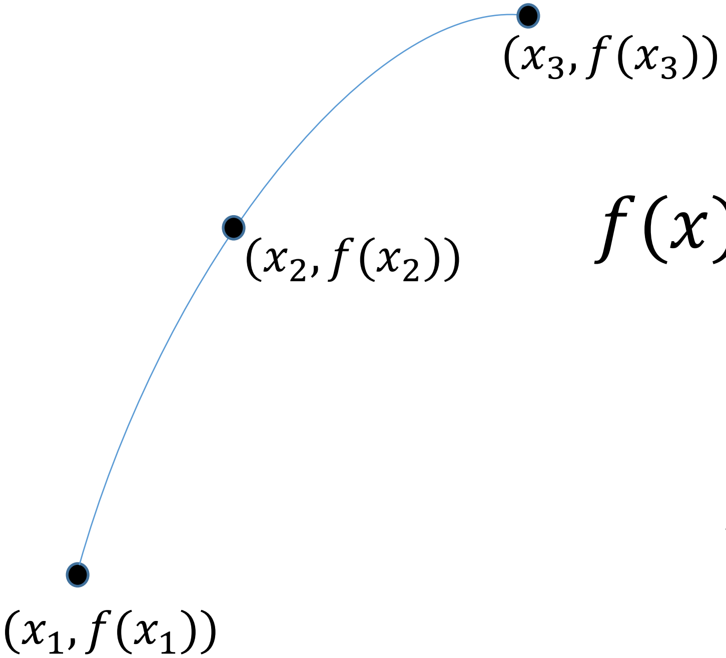
$$f(x_1) = c$$

$$f(x_2) = b(x_2 - x_1) + f(x_1)$$

$$f(x_2) - f(x_1) = b(x_2 - x_1)$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = b$$

Splines



$$f(x) = a(x - x_1)(x - x_2) + b(x - x_1) + c$$

$$f(x_1) = c$$

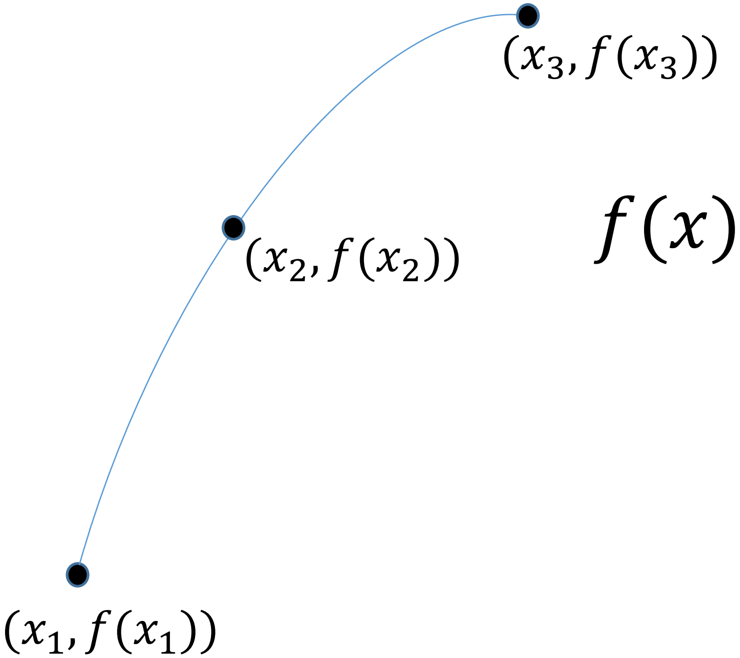
$$f(x_2) = b(x_2 - x_1) + f(x_1)$$

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = b$$

$$\frac{f(x_1 + \Delta x_1) - f(x_1)}{\Delta x_1} = b$$

$$f'(x_1) = b$$

Splines



$$f(x) = a(x - x_1)(x - x_2) + b(x - x_1) + c$$

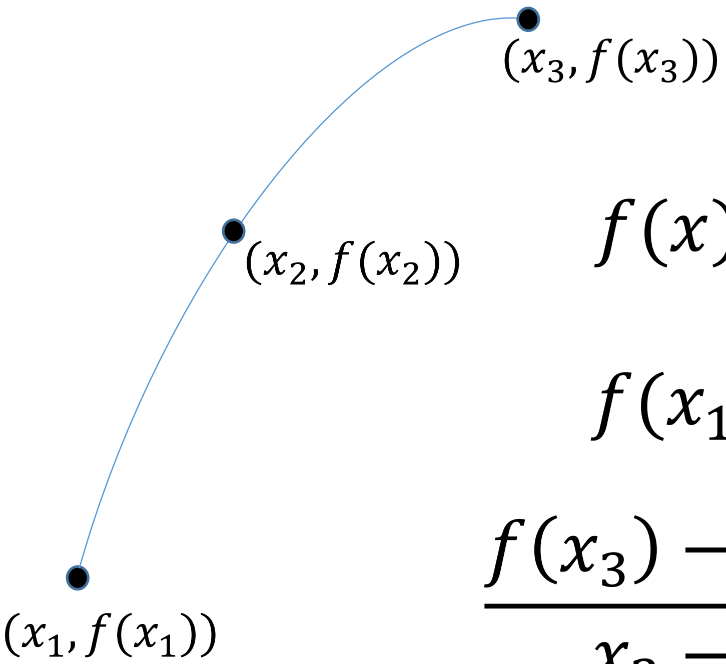
$$f(x_1) = c$$

$$f'(x_1) = b$$

$$f(x_3) = a(x_3 - x_1)(x_3 - x_2) + b(x_3 - x_1) + f(x_1)$$

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} = a(x_3 - x_2) + b$$

Splines



$$f(x) = a(x - x_1)(x - x_2) + b(x - x_1) + c$$

$$f(x_1) = c \qquad\qquad f'(x_1) = b$$

$$\frac{f(x_3) - f(x_1)}{x_3 - x_1} - \frac{f(x_2) - f(x_1)}{x_2 - x_1} = a(x_3 - x_2)$$

$$\frac{\dfrac{f(x_3) - f(x_1)}{x_3 - x_1} - \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}}{x_3 - x_2} = a$$

$$f(x) = a(x - x_1)^2 + b(x - x_1) + c$$

$$f(x_1) = c \qquad f'(x_1) = b$$

$$\frac{f''(x_1)}{2!} = a$$

$$f(x) = f(x_1) + f'(x_1)(x - x_1) + \frac{f''(x_1)}{2!}(x - x_1)^2$$

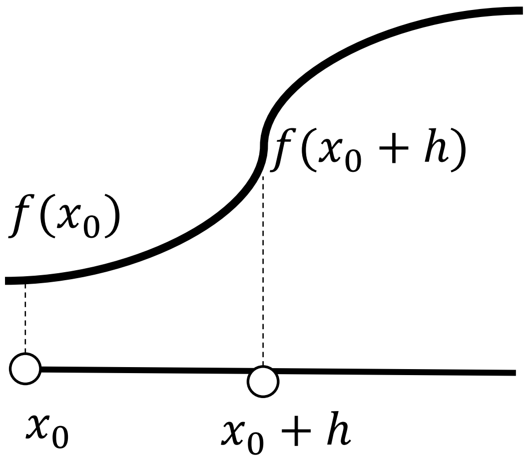
- f is infinitely differentiable about a point x_0

$$f(x) = \sum_{n}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

Taylor's Theorem

$$f(x_0 + h) = \sum_{n}^{N} \frac{f^{(n)}(x_0)h^n}{n!} + R_N(x)$$

Taylor's Approximation



$$f(x_0 + h) = f(x_0) + f'(x_0)h + R_1(x)$$

$$f(x_0 + h) - f(x_0) = f'(x_0)h + R_1(x)$$

Dividing by h

$$\frac{f(x_0 + h)}{h} - \frac{f(x_0)}{h} = f'(x_0) + \frac{R_1(x)}{h}$$

Taylor's Approximation

$$\frac{f(x_0+h)}{h} - \frac{f(x_0)}{h} = f'(x_0) + \frac{R_1(x)}{h}$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} - \frac{R_1(x)}{h}$$

Assume $\frac{R_1(x)}{h}$ *is small, then*

$$f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}$$

Forward Euler Scheme

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\frac{dy}{dt} = g(t, y)$$

$$\frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

$$y_{n+1} = y_n + \Delta t g_n$$

Backward Euler Scheme

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\frac{dy}{dt} = f(t, y)$$

$$\frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

$$y_{n+1} = y_n + \Delta t f_{n+1}$$

Euler Schemes

Explicit Euler Method

$$y_{n+1} = y_n + \Delta t f_n$$

Implicit Euler Method

$$y_{n+1} = y_n + \Delta t f_{n+1}$$

Euler Schemes

$$y' = -ky, y(0) = y_0$$

Explicit Euler Method

$$y_{exact} = y_0 e^{-kt}$$

$$y_{n+1} = y_n - \Delta t k y_n$$

$$y_{n+1} = (1 - \Delta t k) y_n$$

Implicit Euler Method

$$y_{n+1} = y_n - \Delta t k y_{n+1}$$

$$(1 + \Delta t k) y_{n+1} = y_n$$

$$y_{n+1} = \frac{y_n}{(1 + \Delta t k)}$$

Centered Difference

$$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}$$

$$\frac{dy}{dt} = f(t, y)$$

$$\frac{dy}{dt} \approx \frac{y_{n+1} - y_n}{\Delta t}$$

$$y_{n+1} = y_n + \Delta t f_{n+\frac{1}{2}}$$

Taylor's Approximation to solve ODE

Let us solve the following ODE

$$u' = 3u + 2$$

From Taylor's approximation,

$$u'(x_0) \approx \frac{u(x_0 + h) - u(x_0)}{h}$$

$$u' = 3u + 2$$

$$\frac{u(x_0 + h) - u(x_0)}{h} = 3u(x_0) + 2$$

Taylor's Approximation to solve ODE

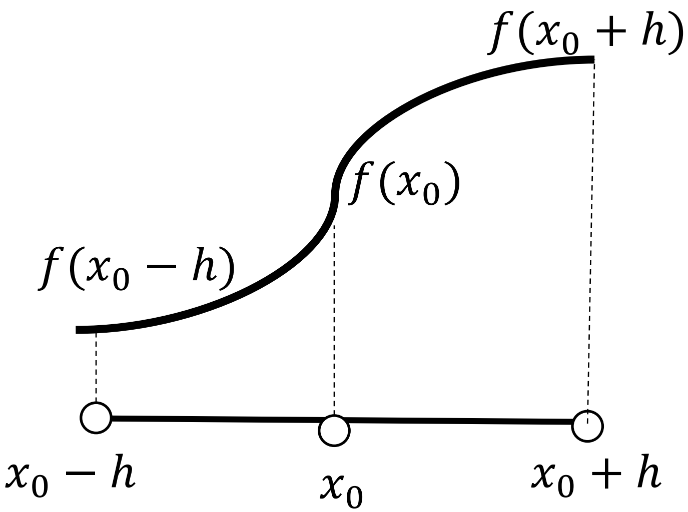
$$u' = 3u + 2$$

$$\frac{u(x_0 + h) - u(x_0)}{h} = 3u(x_0) + 2$$

$$u(x_0 + h) - u(x_0) = 3hu(x_0) + 2h$$

$$u(x_0 + h) = u(x_0) + 3hu(x_0) + 2h$$

Taylor's Approximation 2nd Order



$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{f''(x_0)h^2}{2} + R_2(x)$$

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{f''(x_0)h^2}{2} + R_2^*(x)$$

$$f(x_0+h) + f(x_0-h) \approx 2f(x_0) + f''(x_0)h^2$$

$$f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$

Taylor's Approximation to solve ODE

Let us solve the following ODE

$$u'' = 3u + 2$$

From Taylor's approximation,

$$u''(x_0) \approx \frac{u(x_0 + h) - 2u(x_0) + u(x_0 - h)}{h^2}$$

$$u'' = 3u + 2$$

$$\frac{u(x_0 + h) - 2u(x_0) + u(x_0 - h)}{h^2} = 3u(x_0) + 2$$

Taylor's Approximation to solve ODE

$$\frac{u(x_0+h) - 2u(x_0) + u(x_0-h)}{h^2} = 3u(x_0) + 2$$

Assume $u(x_0) = u_i, u(x_0 - h) = u_{i-1},$

$$u(x_0+h) = u_{i+1}$$

Then

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = 3u_i + 2$$

$$u_{i+1} - 2u_i + u_{i-1} = 3h^2u_i + 2h^2$$

$$u_{i-1} - (2 + 3h^2)u_i + u_{i+1} = 2h^2$$

Taylor's Approximation to solve ODE

$$u_{i-1} - (2 + 3h^2)u_i + u_{i+1} = 2h^2$$

i=1

$$-(2 + 3h^2)u_1 + u_2 = 2h^2$$

i=2

$$u_1 - (2 + 3h^2)u_2 + u_3 = 2h^2$$

i=3

$$u_2 - (2 + 3h^2)u_3 + u_4 = 2h^2$$

i=N

$$u_{N-1} - (2 + 3h^2)u_N = 2h^2$$

Taylor's Approximation to solve ODE

$$-(2+3h^2)u_1 + u_2 = 2h^2$$
$$u_1 - (2+3h^2)u_2 + u_3 = 2h^2$$
$$u_2 - (2+3h^2)u_3 + u_4 = 2h^2$$
$$u_{N-1} - (2+3h^2)u_N = 2h^2$$
$$Au = b$$
$$a = -(2+3h^2)$$

$$A = \begin{bmatrix} a & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & a & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & a & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 & a \end{bmatrix}$$

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 2h^2 \\ 2h^2 \\ \vdots \\ 2h^2 \end{bmatrix}$$

Taylor Series on two Variables

$$z = z_1 + \left(\frac{z_2 - z_1}{x_2 - x_1}\right)(x - x_1) + \left(\frac{z_2 - z_1}{y_2 - y_1}\right)(y - y_1)$$

$$z = f(x, y)$$

$$f(x,y) = f(x_1, y_1) + \left(\frac{f(x_1 + \Delta x_1, y_1) - f(x_1, y_1)}{\Delta x_1}\right)(x - x_1) + \left(\frac{f(x_1, y_1 + \Delta y_1) - f(x_1, y_1)}{\Delta y_1}\right)(y - y_1)$$

Taylor Series on two Variables

$$z = z_1 + \left(\frac{z_2 - z_1}{x_2 - x_1}\right)(x - x_1) + \left(\frac{z_2 - z_1}{y_2 - y_1}\right)(y - y_1)$$

$$z = f(x, y)$$

$$f(x,y) = f(x_1, y_1) + \left(\frac{\partial f(x_1, y_1)}{\partial x}\right)(x - x_1) + \left(\frac{\partial f(x_1, y_1)}{\partial y}\right)(y - y_1)$$

Taylor Series on two Variables

$$f(x,y) = f(x_1,y_1) + \left(\frac{\partial f(x_1,y_1)}{\partial x}\right)(x-x_1) + \left(\frac{\partial f(x_1,y_1)}{\partial y}\right)(y-y_1) + \frac{1}{2!}\left[\left(\frac{\partial^2 f(x_1,y_1)}{\partial x^2}\right)(x-x_1)^2 + \left(\frac{\partial^2 f(x_1,y_1)}{\partial y^2}\right)(x-x_1)(y-y_1) + \left(\frac{\partial^2 f(x_1,y_1)}{\partial y^2}\right)(y-y_1)^2\right]$$

Taylor Series

- f is infinitely differentiable about a point x_0

$$f(x) = \sum_{n}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

Taylor Series on two Variables

$$f(x_1, x_2, \dots, x_d) = \sum_{n_1}^{\infty} \dots \sum_{n_d}^{\infty} \frac{(x_1 - x_0^1)^{n_1} \dots (x_d - x_0^d)^{n_d}}{n_1! \dots n_d!} \left(\frac{\partial^{n_1 + \dots + n_d} f}{\partial x_1^{n_1} \dots \partial x_d^{n_d}} \right) (x_0^1, \dots, x_0^d)$$

$$f(x, y) = \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \frac{(x - x_0)^{n_1} (y - y_0)^{n_2}}{n_1! \; n_2!} \left(\frac{\partial^{n_1 + n_2} f}{\partial x^{n_1} \partial y^{n_2}} \right) (x_0, y_0)$$

1D Heat Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \begin{matrix} 0 < x < l, \\ 0 < t < \infty \end{matrix}$$

$$T(0,t) = f(t), 0 < t < \infty$$

$$T(l,t) = g(t),\ 0 < t < \infty$$

$$T(x,0) = s(x), 0 \leq x \leq l$$

α-diffusion coefficient

Diffusion:
behavior of the collective motion of micro-particles in a material resulting from the random movement of each micro-particle.

$T(x,0) = s(x)$

$T(0,t) = f(t)$  $T(l,t) = g(t)$

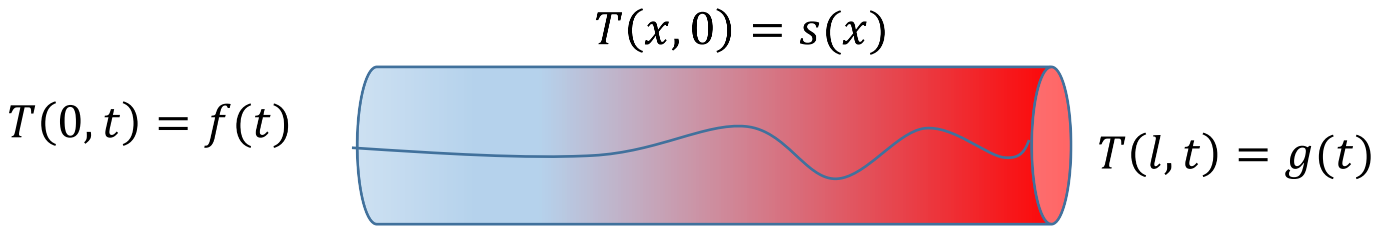
1D Heat Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} \qquad \begin{array}{l} 0 < x < l, \\ 0 < t < \infty \end{array}$$

$$T(0,t) = f(t), 0 < t < \infty$$

$$T(l,t) = g(t),\ 0 < t < \infty$$

$$T(x,0) = s(x), 0 < x < l$$



1D Heat Equation

- **Diffusion equation converges to a stationary solution $T_s(x)$ as $t \to \infty$**
- **In this limit $T_t = 0$ and $T_s''(x) = 0$**
- **This stationary limit of the diffusion equation is called Laplace equation**

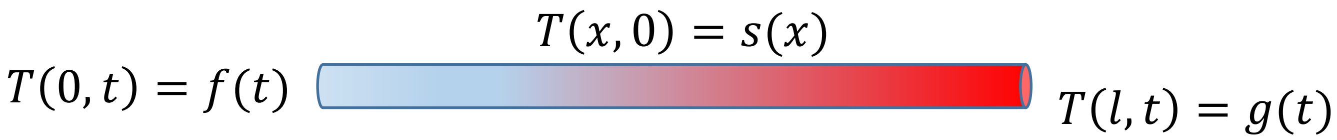
1D Heat Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad \begin{matrix} 0 < x < l, \\ 0 < t < \infty \end{matrix}$$

$$T(0,t) = 0, 0 < t < \infty$$

$$T(l,t) = 0, 0 < t < \infty$$

$$T(x,0) = s(x), 0 \leq x \leq l$$



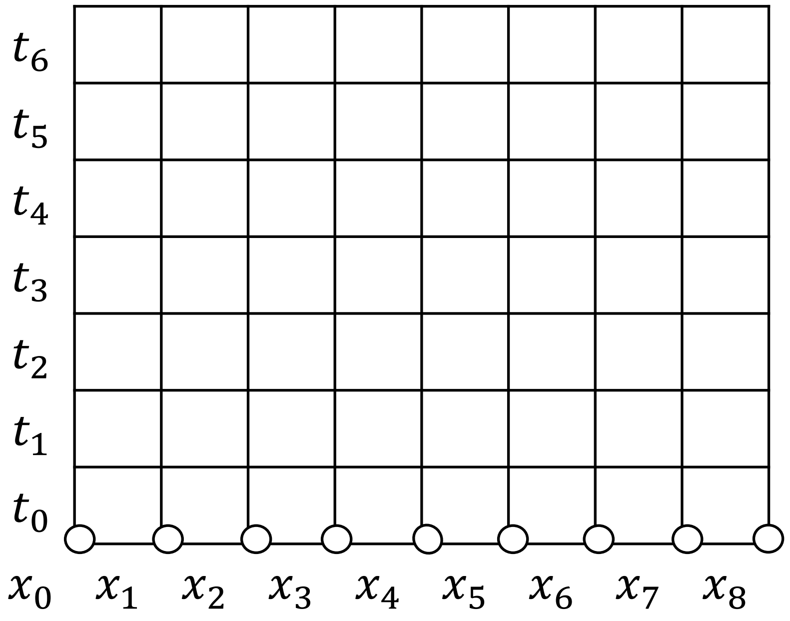
Forward Euler Scheme

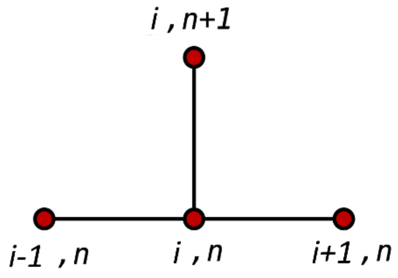
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$x_i = i\Delta x, i = 0,1,2, \ldots, N_x$$

$$t_n = n\Delta t, n = 0,1,2, \ldots, N_t$$

$$T(x_i, t_n) = T_i^n$$

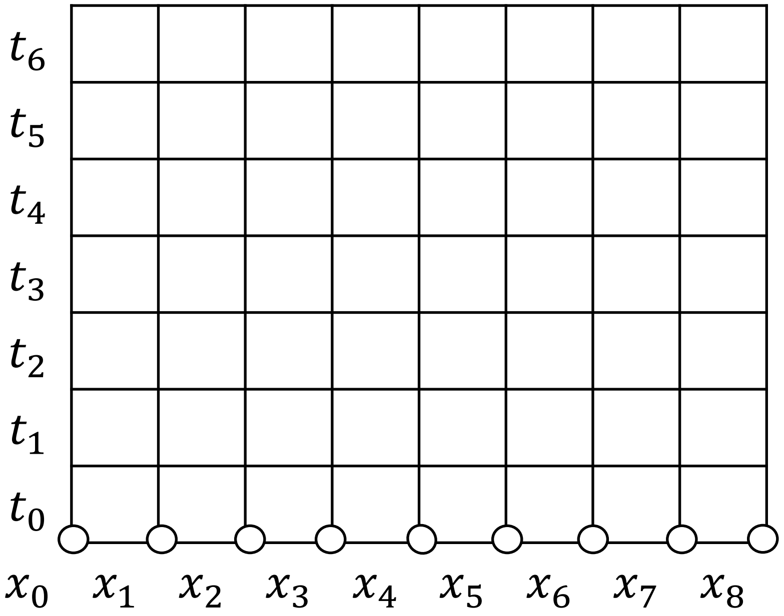




Forward Euler Scheme

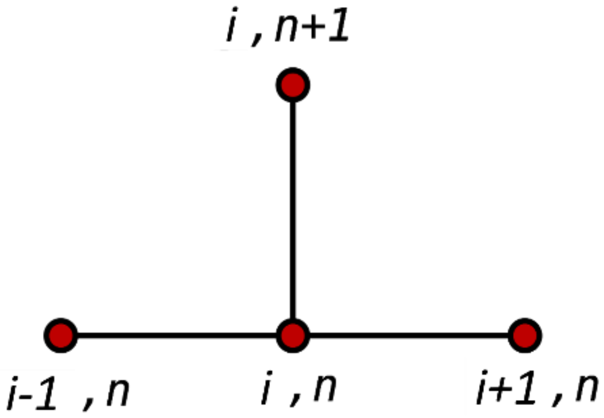
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$

$$T(x_i, t_n) = T_i^n$$



$$\frac{\partial T(x_i, t_n)}{\partial t} = \alpha \frac{\partial^2 T(x_i, t_n)}{\partial x^2} + f(x_i, t_n)$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + f_i^n$$



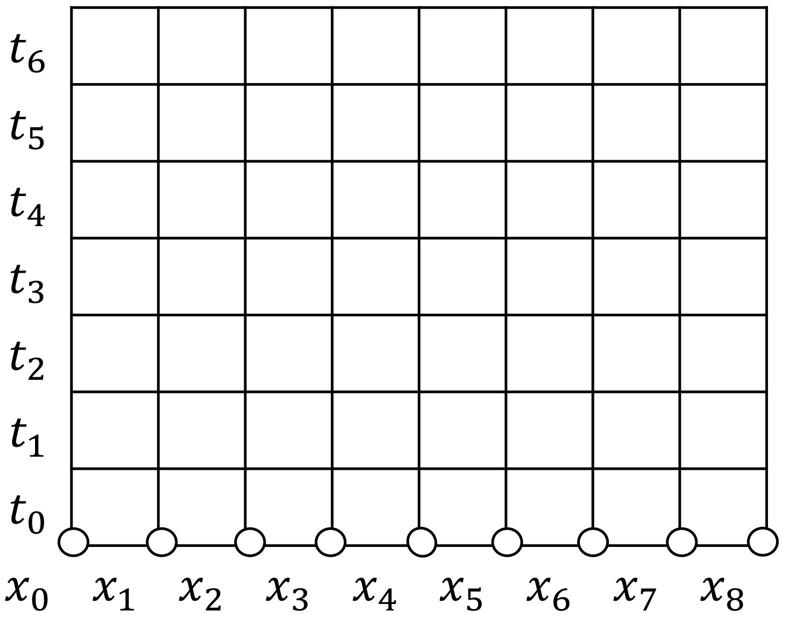
Forward Euler Scheme

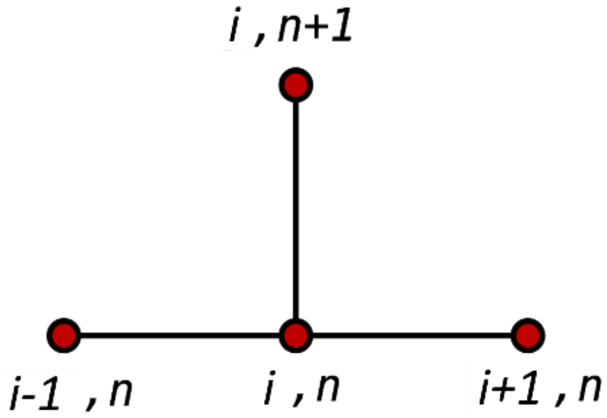
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$T(x_i, t_n) = T_i^n$$

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + f_i^n$$

$$T_i^{n+1} = T_i^n + \alpha \frac{\Delta t}{\Delta x^2}(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + f_i^n \Delta t$$

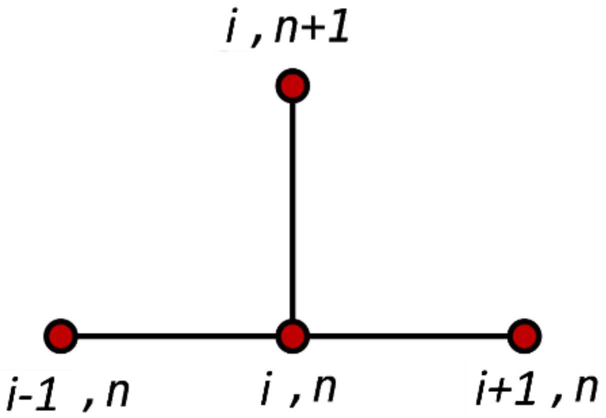


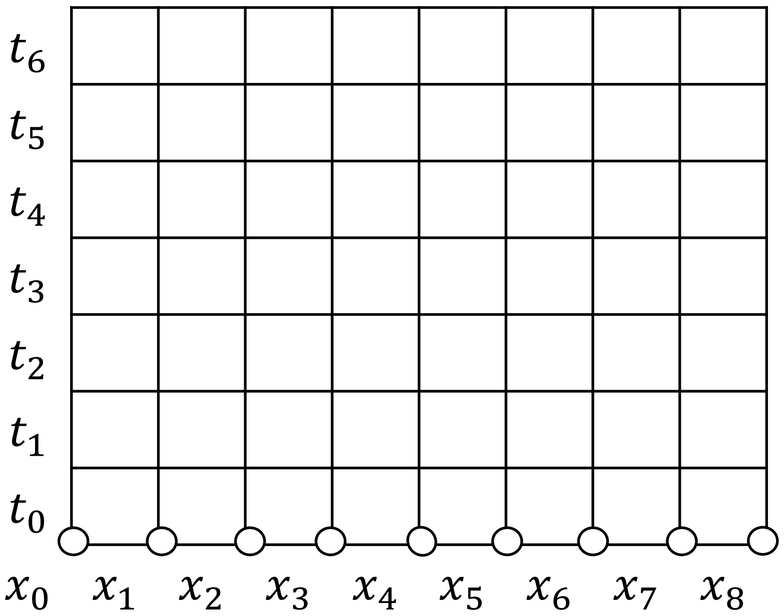


Forward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$T(x_i, t_n) = T_i^n$$





$$T_i^{n+1} = T_i^n + F(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + f_i^n \Delta t$$

$$F = \alpha \frac{\Delta t}{\Delta x^2}$$

F is the dimensionless number that lumps the key physical parameter in the problem, α, and the discretization parameters Δx and Δt into a single parameter. Depending on F, the numerical method schemes are chosen.

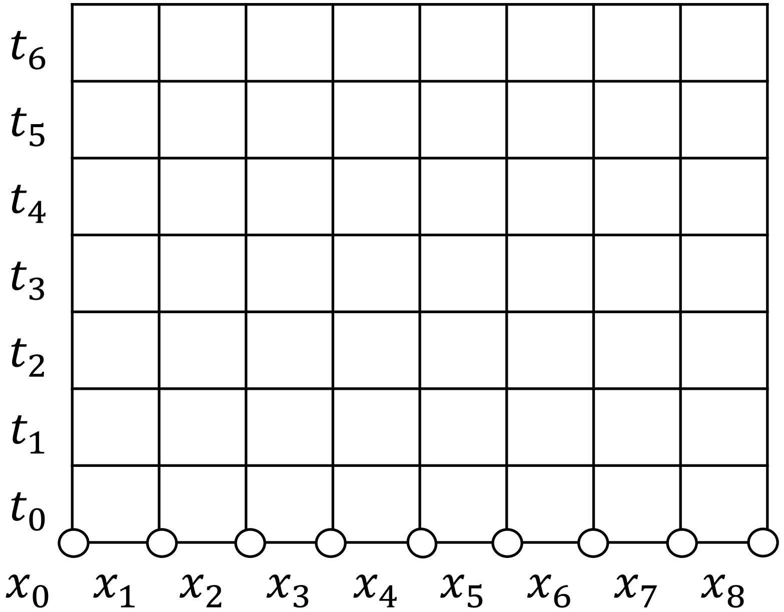
Backward Euler Scheme

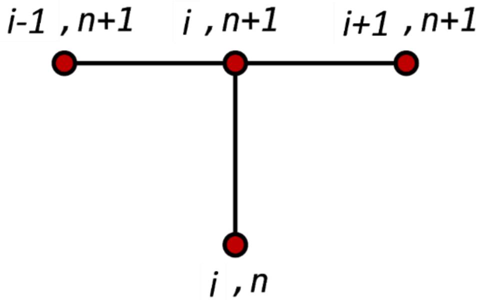
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$x_i = i\Delta x, i = 0,1,2, \dots, N_x$$

$$t_n = n\Delta t, n = 0,1,2, \dots, N_t$$

$$T(x_i, t_n) = T_i^n$$

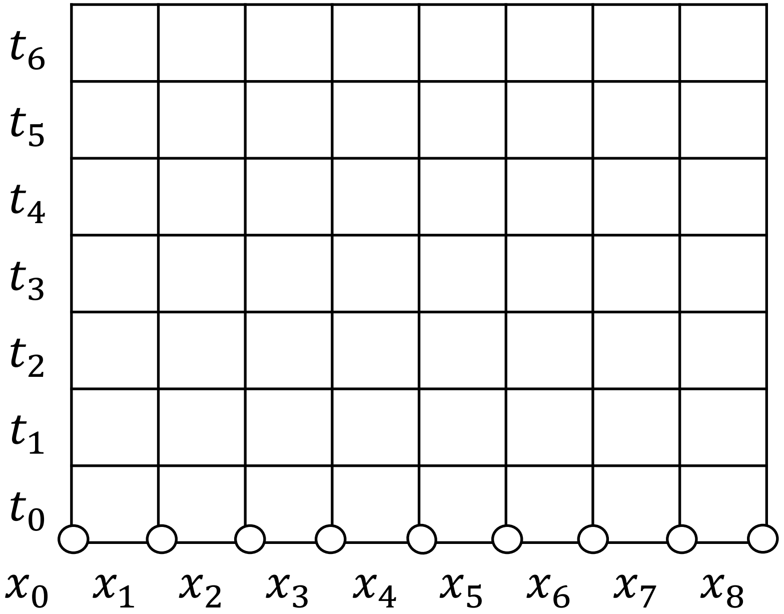




Backward Euler Scheme

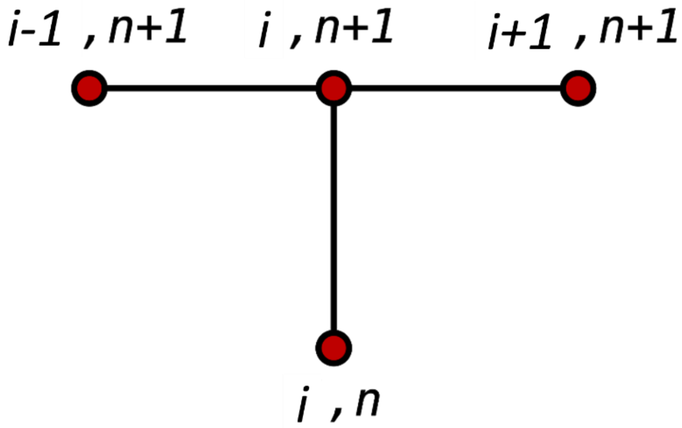
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f$$

$$f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$



$$T(x_i, t_n) = T_i^n$$

$$\frac{\partial T(x_i, t_n)}{\partial t} = \alpha \frac{\partial^2 T(x_i, t_n)}{\partial x^2} + f(x_i, t_n)$$



$$\frac{T_i^n - T_i^{n-1}}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + f_i^n$$

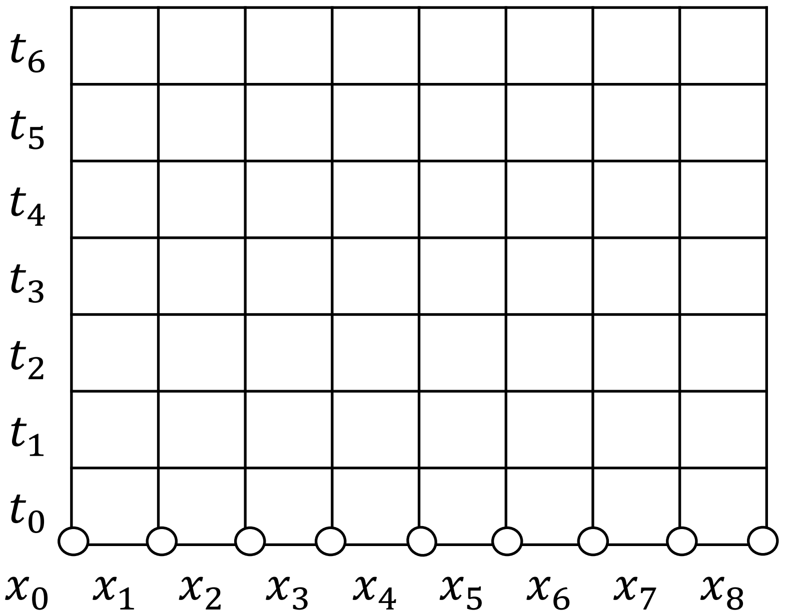
Backward Euler Scheme

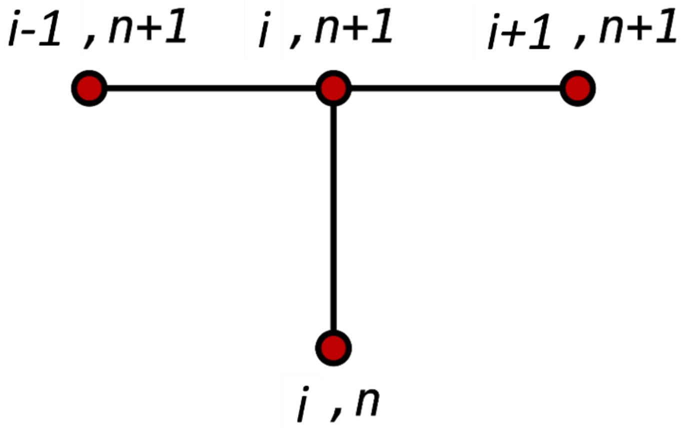
$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad T(x_i, t_n) = T_i^n$$

$$\frac{T_i^n - T_i^{n-1}}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{\Delta x^2} + f_i^n$$

$$T_i^n - T_i^{n-1} = \alpha \left(\frac{\Delta t}{\Delta x^2}\right)(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t f_i^n$$

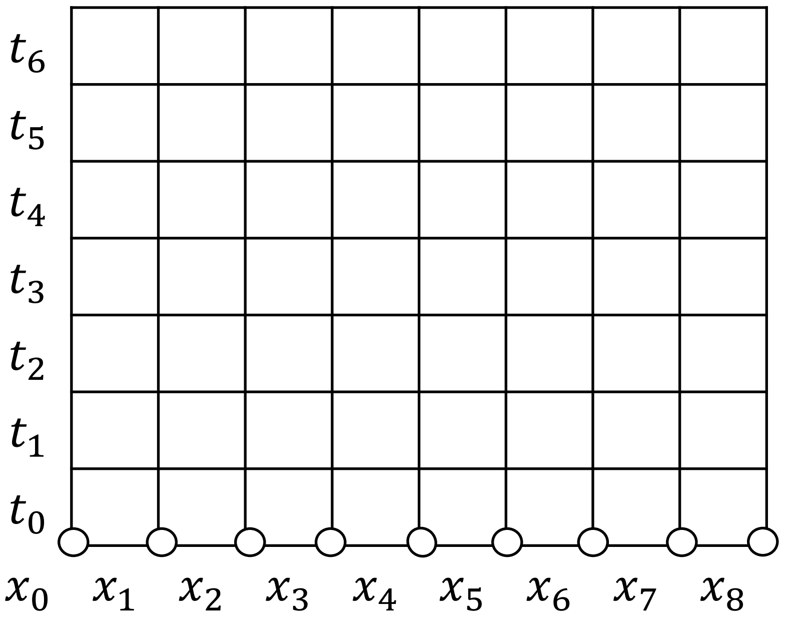
$$T_i^n - T_i^{n-1} = F(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t f_i^n$$





Backward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad T(x_i, t_n) = T_i^n$$



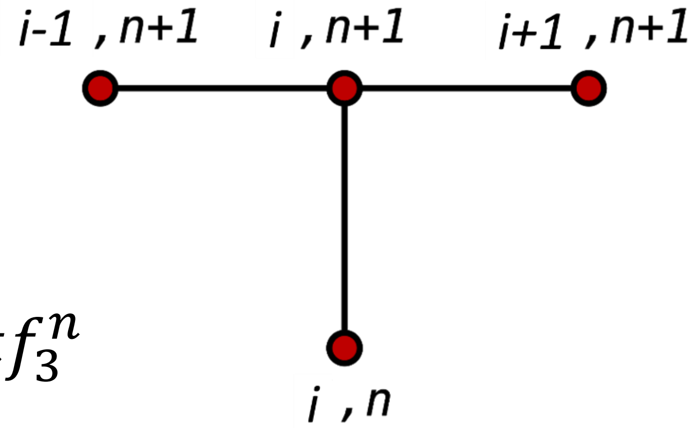
$$T_i^n - T_i^{n-1} = F(T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t f_i^n$$

$$-FT_{i-1}^n + (1+2F)T_i^n - FT_{i+1}^n = T_{i-1}^{n-1} + \Delta t f_i^n$$

$$-FT_0^n + (1+2F)T_1^n - FT_2^n = T_1^{n-1} + \Delta t f_1^n$$

$$-FT_1^n + (1+2F)T_2^n - FT_3^n = T_2^{n-1} + \Delta t f_2^n$$

$$-FT_2^n + (1+2F)T_3^n - FT_4^n = T_3^{n-1} + \Delta t f_3^n$$



Backward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad T(x_i, t_n) = T_i^n$$

$$-FT_{i-1}^n + (1 + 2F)T_i^n - FT_{i+1}^n = T_{i-1}^{n-1} + \Delta t f_i^n$$

$$AT = b$$

$$A = \begin{bmatrix} a_{00} & a_{01} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ a_{10} & a_{11} & a_{12} & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ 0 & a_{21} & a_{22} & a_{23} & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & 0 & a_{32} & a_{33} & a_{34} & 0 & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & 0 & a_{ii-1} & a_{ii} & a_{ii+1} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & a_{N_x-1N_x} \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a_{N_xN_x-1} & a_{N_xN_x} \end{bmatrix}$$

$$a_{ii-1} = a_{ii+1} = -F$$

$$a_{ii} = 1 + 2F$$

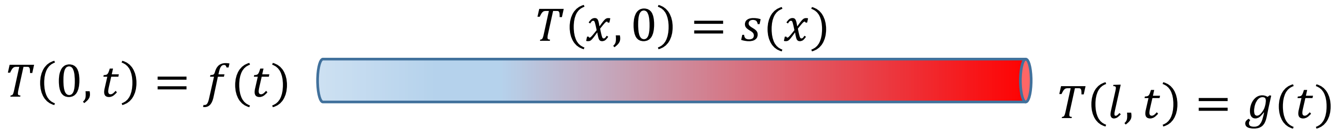
1D Heat Equation

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad \begin{array}{l} 0 < x < l, \\ 0 < t < \infty \end{array}$$

$$T(0,t) = 0, 0 < t < \infty$$

$$T(l,t) = 0, 0 < t < \infty$$

$$T(x,0) = s(x), 0 \le x \le l$$



Backward Euler Scheme

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} + f \qquad T(x_i, t_n) = T_i^n$$

$$-FT_{i-1}^n + (1+2F)T_i^n - FT_{i+1}^n = T_{i-1}^{n-1} + \Delta t f_i^n$$

$$AT^n = b \qquad a_{ii-1} = a_{ii+1} = -F \qquad a_{ii} = 1 + 2F$$

$$A = \begin{bmatrix}
a_{00} & a_{01} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
a_{10} & a_{11} & a_{12} & \ddots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & a_{21} & a_{22} & a_{23} & \ddots & \cdots & \cdots & \cdots & \cdots & \vdots \\
\vdots & 0 & a_{32} & a_{33} & a_{34} & 0 & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \cdots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots \\
\vdots & \vdots & \cdots & 0 & a_{ii-1} & a_{ii} & a_{ii+1} & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & a_{N_x-1N_x} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & a_{N_xN_x-1} & a_{N_xN_x}
\end{bmatrix}$$

$$b_0 = b_{N_x} = 0, b_i = T_i^{n-1}$$

$$T^n = \begin{bmatrix} T_0^n \\ T_1^n \\ T_2^n \\ \vdots \\ T_{N_x-1}^n \\ T_{N_x}^n \end{bmatrix} \qquad b = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_{N_x-1} \\ b_{N_x} \end{bmatrix}$$

Taylor Series on two Variables

$$z = z_1 + \left(\frac{z_2 - z_1}{x_2 - x_1}\right)(x - x_1) + \left(\frac{z_2 - z_1}{y_2 - y_1}\right)(y - y_1)$$

$$z = f(x, y)$$

$$\begin{aligned} f(x, y) \\ = f(x_1, y_1) + \left(\frac{f(x_1 + \Delta x_1, y_1) - f(x_1, y_1)}{\Delta x_1}\right)(x - x_1) \\ + \left(\frac{f(x_1, y_1 + \Delta y_1) - f(x_1, y_1)}{\Delta y_1}\right)(y - y_1) \end{aligned}$$

Taylor Series on two Variables

$$z = z_1 + \left(\frac{z_2 - z_1}{x_2 - x_1}\right)(x - x_1) + \left(\frac{z_2 - z_1}{y_2 - y_1}\right)(y - y_1)$$

$$z = f(x, y)$$

$$f(x, y) = f(x_1, y_1) + \left(\frac{\partial f(x_1, y_1)}{\partial x}\right)(x - x_1) + \left(\frac{\partial f(x_1, y_1)}{\partial y}\right)(y - y_1)$$

Taylor Series on two Variables

$$
\begin{aligned}
f(x,y) &= f(x_1,y_1) + \left(\frac{\partial f(x_1,y_1)}{\partial x}\right)(x-x_1) + \left(\frac{\partial f(x_1,y_1)}{\partial y}\right)(y-y_1) \\
&+ \frac{1}{2!}\left[\left(\frac{\partial^2 f(x_1,y_1)}{\partial x^2}\right)(x-x_1)^2 + \left(\frac{\partial^2 f(x_1,y_1)}{\partial y^2}\right)(x-x_1)(y-y_1) \right.\\
&\left. + \left(\frac{\partial^2 f(x_1,y_1)}{\partial y^2}\right)(y-y_1)^2\right]
\end{aligned}
$$

Taylor Series

- f is infinitely differentiable about a point x_0

$$f(x) = \sum_{n}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}$$

Taylor Series on two Variables

$$f(x_1, x_2, \dots, x_d) = \sum_{n_1}^{\infty} \dots \sum_{n_d}^{\infty} \frac{(x_1 - x_0^1)^{n_1} \dots \left(x_d - x_0^d\right)^{n_d}}{n_1! \dots n_d!} \left(\frac{\partial^{n_1+\dots+n_d} f}{\partial x_1^{n_1} \dots \partial x_d^{n_d}} \right) (x_0^1, \dots, x_0^d)$$

$$f(x, y) = \sum_{n_1}^{\infty} \sum_{n_2}^{\infty} \frac{(x - x_0)^{n_1} (y - y_0)^{n_2}}{n_1!\; n_2!} \left(\frac{\partial^{n_1+n_2} f}{\partial x^{n_1} \partial y^{n_2}} \right) (x_0, y_0)$$

2D Heat Equation

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \quad \begin{array}{l} 0 < x < l_1, 0 < y < l_2 \\ 0 < t < \infty \end{array}$$

$$T(x, 0, t) = f_1(t), 0 < t < \infty \qquad T(l_1, y, t) = f_2(t), 0 < t < \infty$$

$$T(x, l_2, t) = f_3(t), 0 < t < \infty \qquad T(0, y, t) = f_4(t), 0 < t < \infty$$

$$T(x, y, 0) = s(x, y), 0 \leq x \leq l_1, 0 \leq y \leq l_2$$

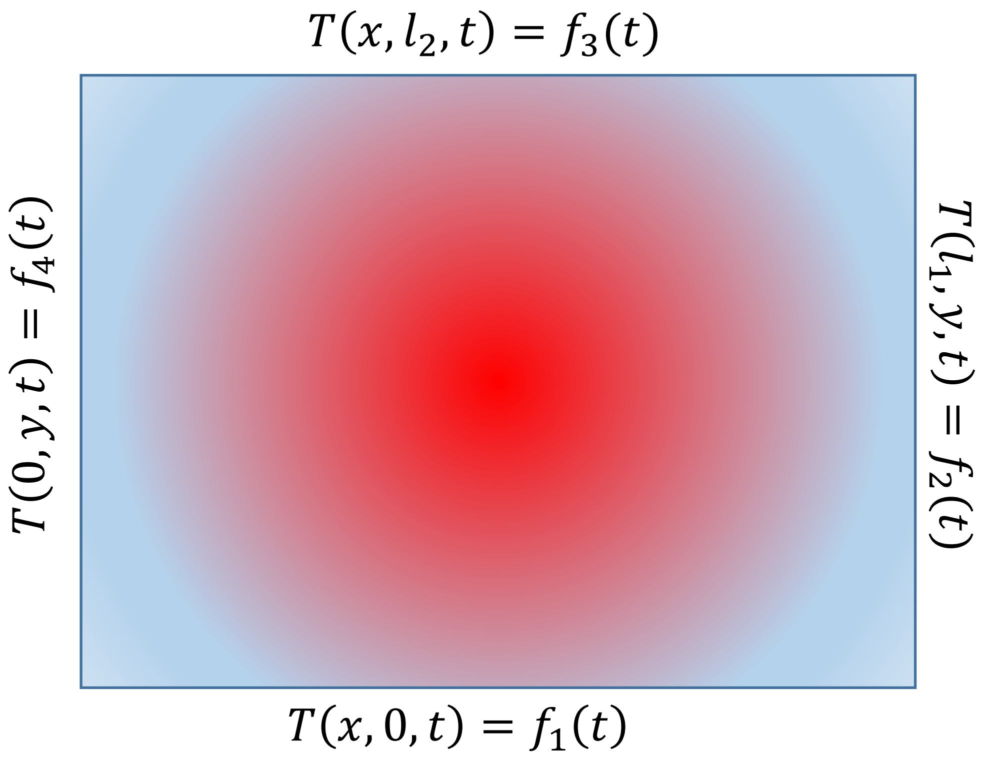
1D Heat Equation

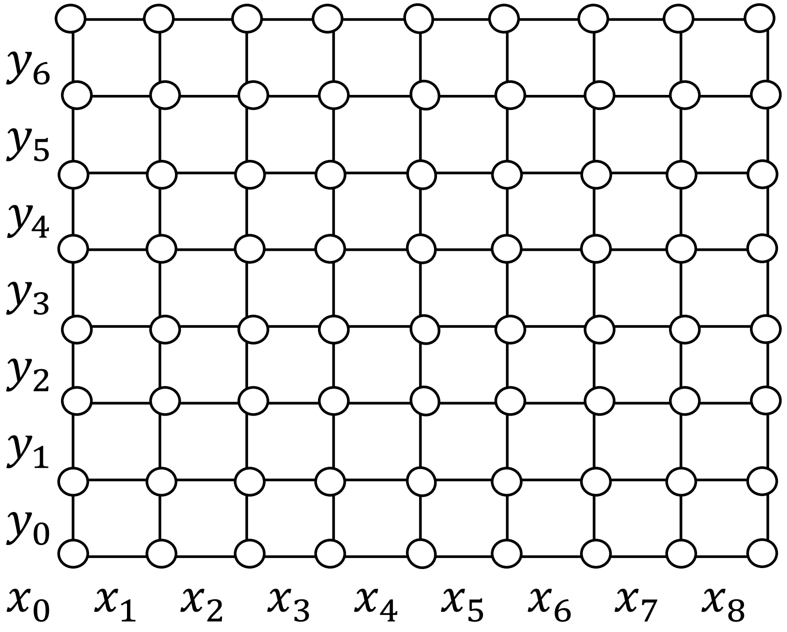
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad 0 < x < l_1, 0 < y < l_2\ 0 < t < \infty$$

$T(x,0,t) = f_1(t), 0 < t < \infty \qquad T(l_1,y,t) = f_2(t),\ 0 < t < \infty$

$T(x,l_2,t) = f_3(t),\ 0 < t < \infty \qquad T(0,y,t) = f_4(t), 0 < t < \infty$

$T(x,y,0) = s(x,y), 0 \le x \le l_1, 0 \le y \le l_2$



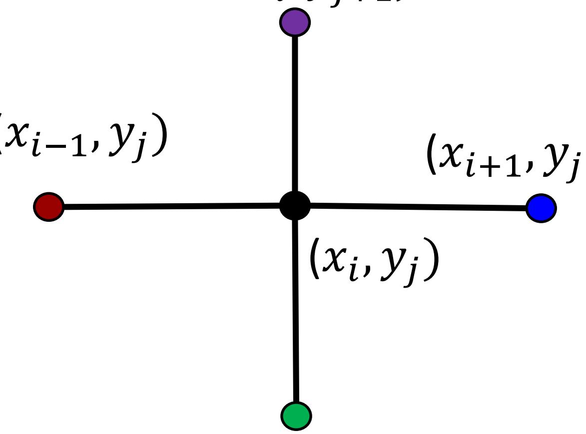


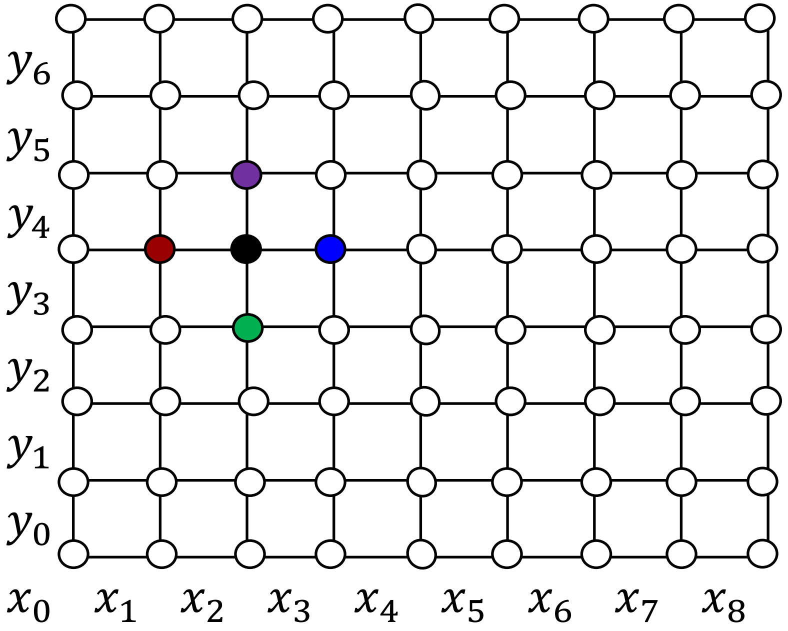
Forward Euler Scheme

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$x_i = i\Delta x, i = 0,1,2,\dots,N_x$$
$$y_j = \mathrm{j}\Delta y, j = 0,1,2,\dots,N_y$$
$$t_n = n\Delta t, n = 0,1,2,\dots,N_t$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$



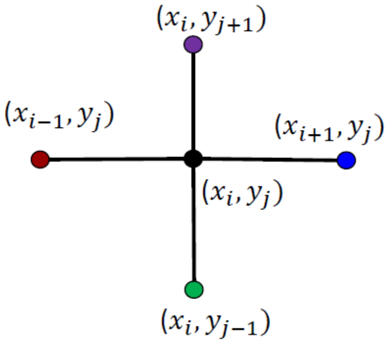


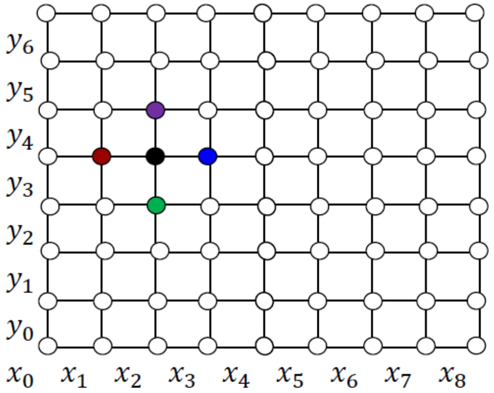
$$\frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t} = \left(\frac{T_{i+1j}^n - 2T_{ij}^n + T_{i-1j}^n}{\Delta x^2}\right) + \left(\frac{T_{ij+1}^n - 2T_{ij}^n + T_{ij-1}^n}{\Delta y^2}\right)$$

Taylor Series: $f(x,y) = \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} \frac{(x-x_0)^{n_1}(y-y_0)^{n_2}}{n_1!\,n_2!}\left(\frac{\partial^{n_1+n_2} f}{\partial x^{n_1}\partial y^{n_2}}\right)(x_0, y_0)$

Taylor Series: $f(x,y) = \sum_{n_1}^{\infty}\sum_{n_2}^{\infty} \frac{(x-x_0)^{n_1}(y-y_0)^{n_2}}{n_1!\,n_2!}\left(\frac{\partial^{n_1+n_2} f}{\partial x^{n_1}\partial y^{n_2}}\right)(x_0,y_0)$

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$





$$x_i = i\Delta x, i = 0,1,2,\dots,N_x$$
$$y_j = \mathrm{j}\Delta y, j = 0,1,2,\dots,N_y$$
$$t_n = n\Delta t, n = 0,1,2,\dots,N_t$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$

$$\frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t} = \left(\frac{T_{i+1j}^n - 2T_{ij}^n + T_{i-1j}^n}{\Delta x^2}\right) + \left(\frac{T_{ij+1}^n - 2T_{ij}^n + T_{ij-1}^n}{\Delta y^2}\right)$$

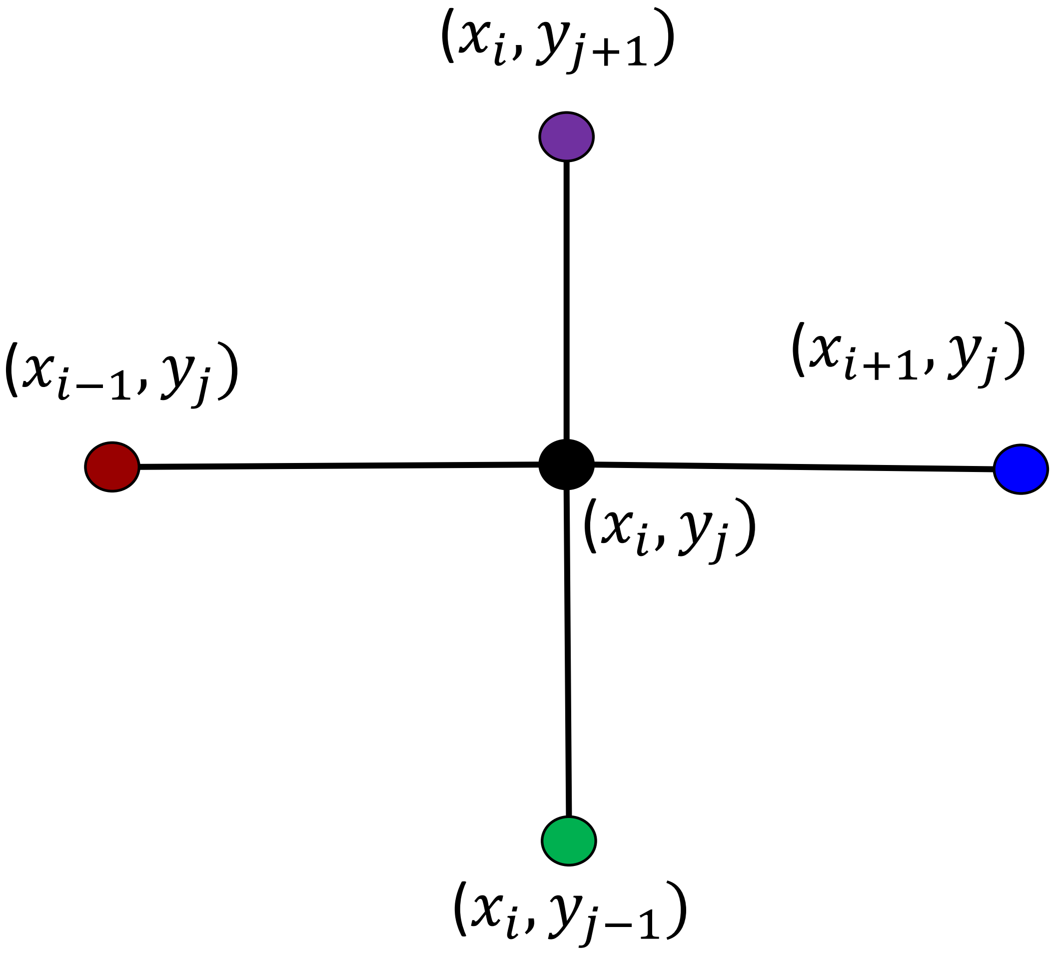
$$T_{ij}^{n+1} = \left(1 - \frac{4k\Delta t}{h^2}\right)T_{ij}^n + k\Delta t\left(\frac{T_{i-1j}^n + T_{ij-1}^n + T_{i+1j}^n + T_{ij+1}^n}{h^2}\right)$$

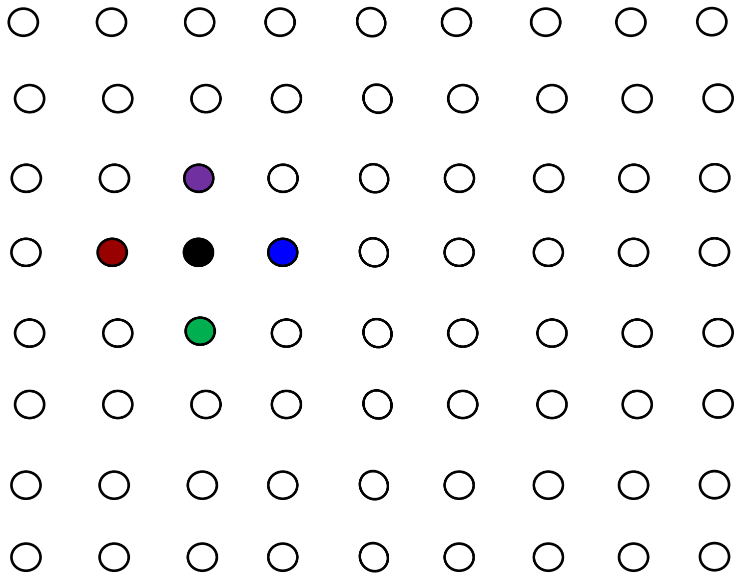
$$\frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t} = \left(\frac{T_{i+1j}^{n+1} - 2T_{ij}^{n+1} + T_{i-1j}^{n+1}}{\Delta x^2}\right) + \left(\frac{T_{ij+1}^{n+1} - 2T_{ij}^{n+1} + T_{ij-1}^{n+1}}{\Delta y^2}\right)$$

$$AT^{n+1} = b$$

$$A = \begin{bmatrix}
1 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
0 & 1 & 0 & \ddots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\
\vdots & -F & 0 & \dots & -F & 1+4F & \dots & -F & \dots & \dots & -F & \dots & \vdots \\
\vdots & \vdots & -F & \ddots & \ddots & -F & 1+4F & \dots & -F & \dots & \dots & \dots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & -F & \ddots & \dots & \dots & \dots & \dots & \dots & \vdots \\
\vdots & \vdots & \dots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & 0 \\
0 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \dots & \dots & \dots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \dots & \dots & \dots & 1
\end{bmatrix}
\quad T^{n+1} = \begin{bmatrix}
T_{00}^{n+1} \\ T_{01}^{n+1} \\ \vdots \\ T_{0N_y-1}^{n+1} \\ T_{10}^{n+1} \\ T_{11}^{n+1} \\ \vdots \\ T_{1N_y-1}^{n+1} \\ \vdots \\ \vdots \\ T_{N_x-10}^{n+1} \\ T_{N_x-11}^{n+1} \\ \vdots \\ T_{N_x-1N_y-1}^{n+1}
\end{bmatrix}
\quad b = \begin{bmatrix}
T_b \\ T_b \\ \vdots \\ T_b \\ T_l \\ T_{11}^n \\ \vdots \\ T_{1N_y-1}^n \\ \vdots \\ \vdots \\ T_u \\ T_u \\ \vdots \\ T_u
\end{bmatrix}$$

Forward Euler Scheme





Forward Euler Scheme

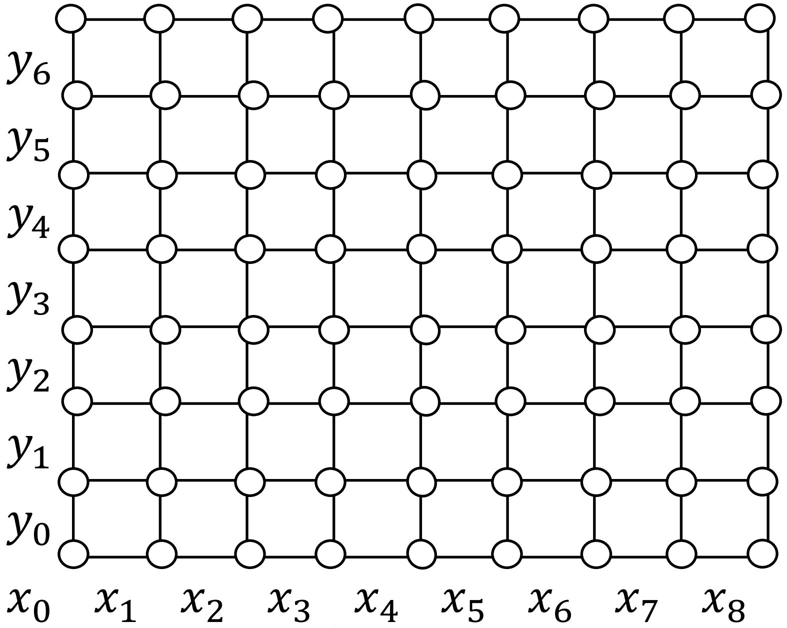
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

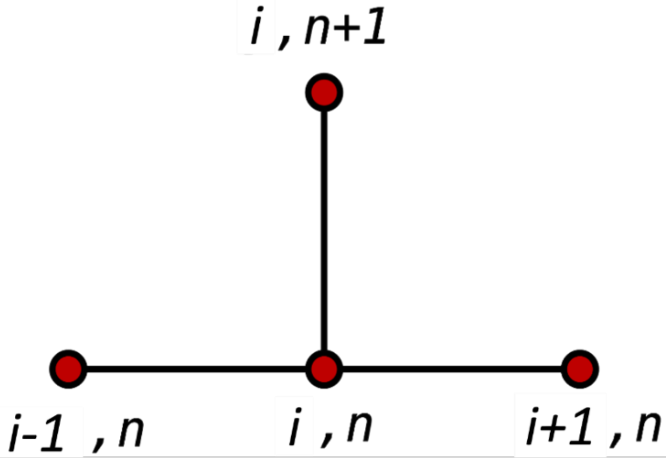
$$f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$

$$\frac{\partial T(x_i, y_j, t_n)}{\partial t} = k\left(\frac{\partial^2 T(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 T(x_i, y_j, t_n)}{\partial y^2}\right)$$

$$\frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t} = \left(\frac{T_{i+1j}^n - 2T_{ij}^n + T_{i-1j}^n}{\Delta x^2}\right) + \left(\frac{T_{ij+1}^n - 2T_{ij}^n + T_{ij-1}^n}{\Delta y^2}\right)$$





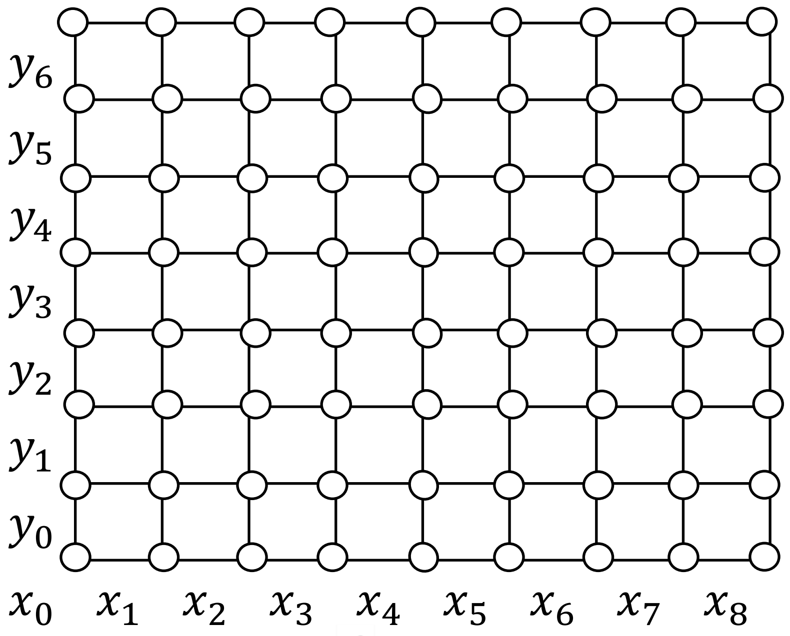
Forward Euler Scheme

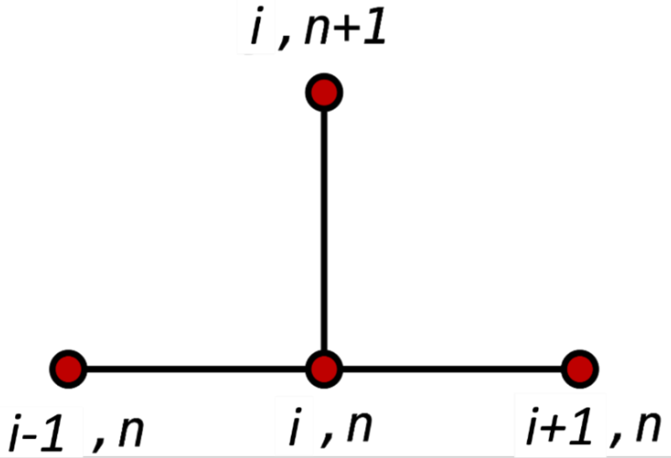
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\Delta x = \Delta y = h$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$

$$\frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t} = k\left(\frac{T_{i-1j}^n + T_{ij-1}^n - 4T_{ij}^n + T_{i+1j}^n + T_{ij+1}^n}{h^2}\right)$$





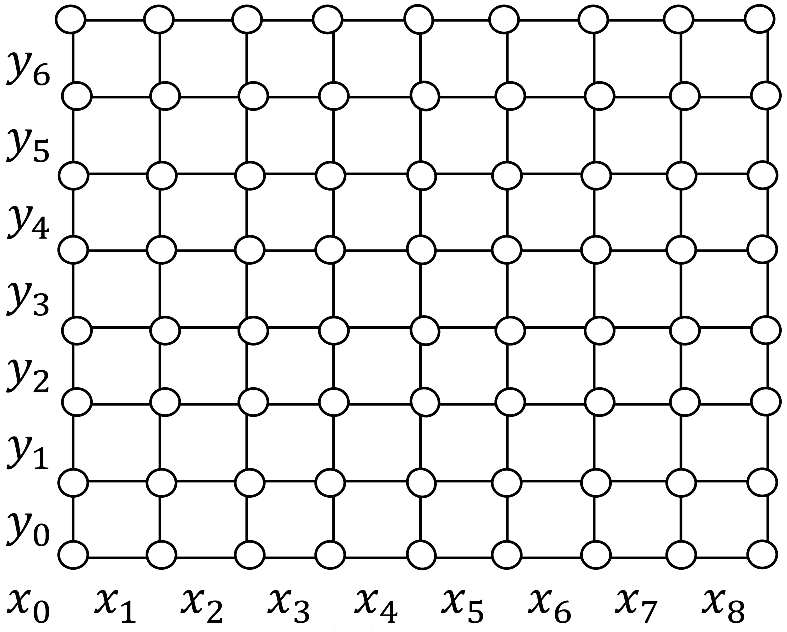
Forward Euler Scheme

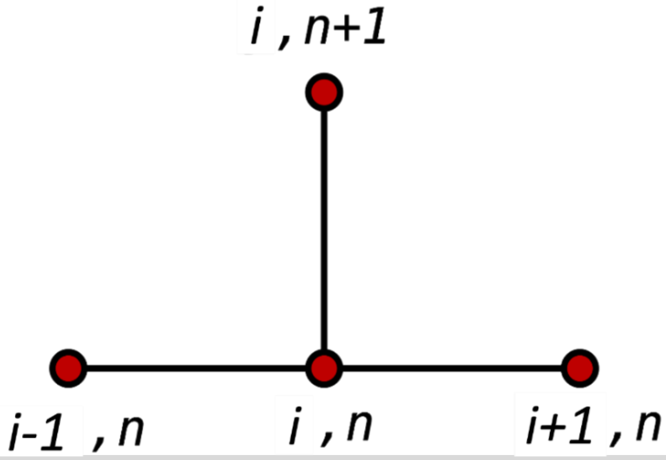
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\Delta x = \Delta y = h$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$

$$T_{ij}^{n+1} = T_{ij}^n + k\Delta t\left(\frac{T_{i-1j}^n + T_{ij-1}^n - 4T_{ij}^n + T_{i+1j}^n + T_{ij+1}^n}{h^2}\right)$$





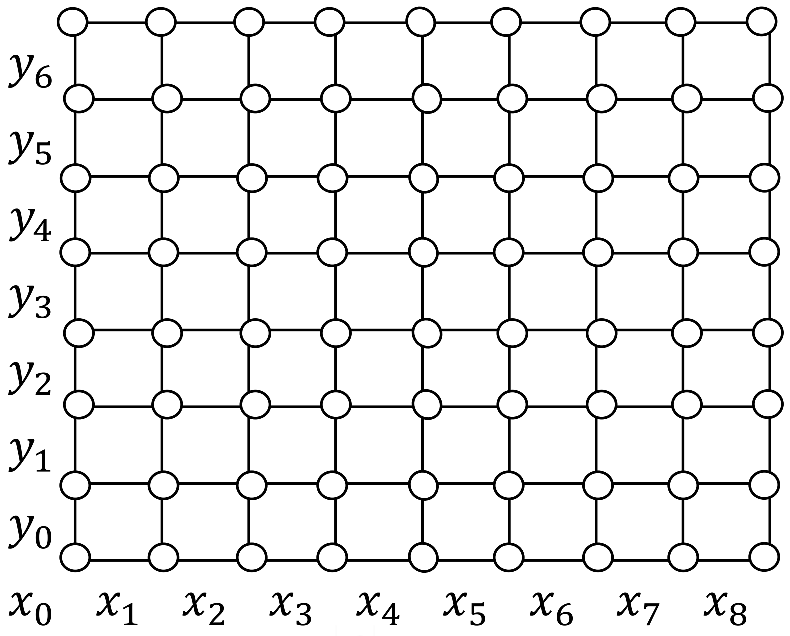
Forward Euler Scheme

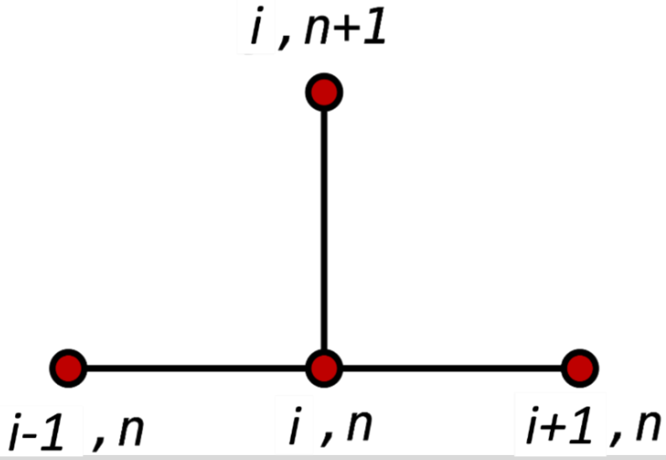
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\Delta x = \Delta y = h$$

$$T(x_i, y_j, t_n) = T_{ij}^{n}$$

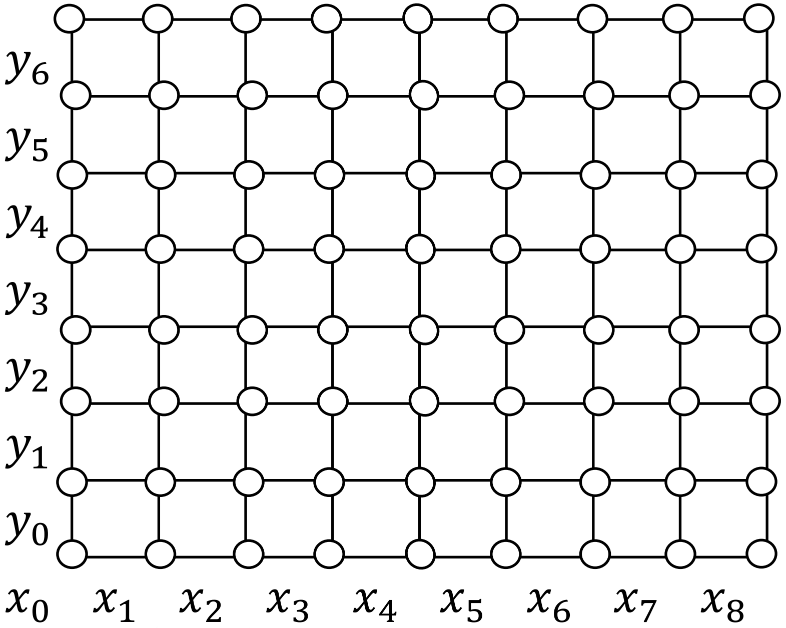
$$T_{ij}^{n+1} = \left(1 - \frac{4k\Delta t}{h^2}\right)T_{ij}^{n} + k\Delta t\left(\frac{T_{i-1j}^{n} + T_{ij-1}^{n} + T_{i+1j}^{n} + T_{ij+1}^{n}}{h^2}\right)$$





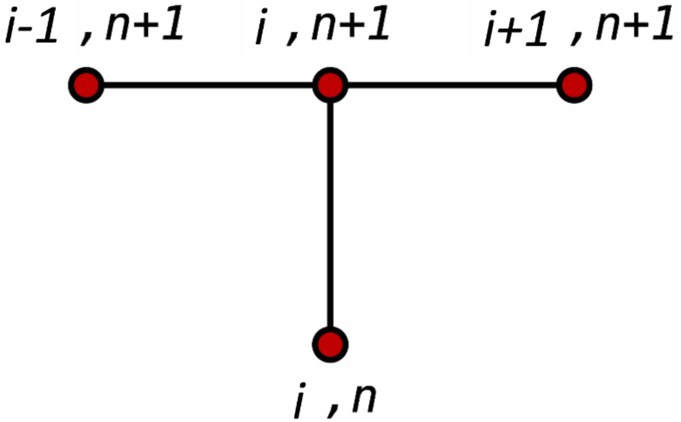
Backward Euler Scheme

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad f''(x_0) \approx \frac{f(x_0+h) - 2f(x_0) + f(x_0-h)}{h^2}$$



$$T(x_i, y_j, t_n) = T_{ij}^n$$

$$\frac{\partial T(x_i, y_j, t_n)}{\partial t} = k\left(\frac{\partial^2 T(x_i, y_j, t_n)}{\partial x^2} + \frac{\partial^2 T(x_i, y_j, t_n)}{\partial y^2}\right)$$



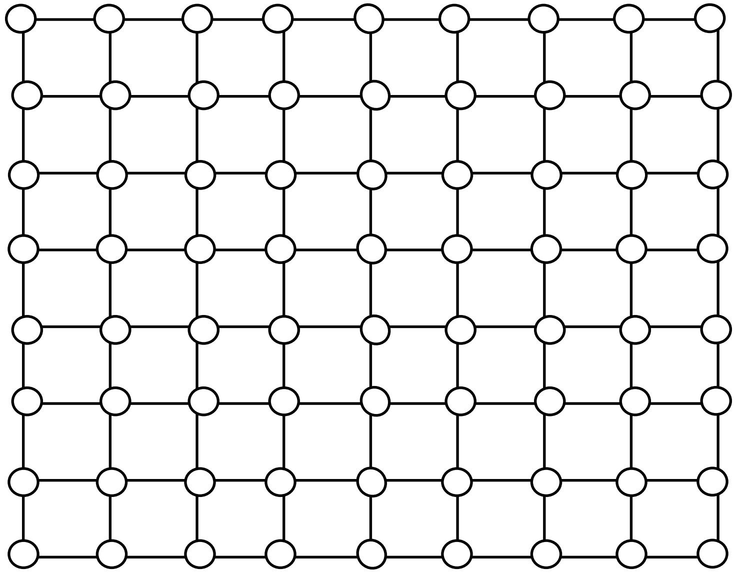
$$\frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t} = \left(\frac{T_{i+1j}^{n+1} - 2T_{ij}^{n+1} + T_{i-1j}^{n+1}}{\Delta x^2}\right) + \left(\frac{T_{ij+1}^{n+1} - 2T_{ij}^{n+1} + T_{ij-1}^{n+1}}{\Delta y^2}\right)$$

Backward Euler Scheme

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\Delta x = \Delta y = h$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$



$$\frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t} = k\left(\frac{T_{i-1j}^{n+1} + T_{ij-1}^{n+1} - 4T_{ij}^{n+1} + T_{i+1j}^{n+1} + T_{ij+1}^{n+1}}{h^2}\right)$$

$$\frac{T_{ij}^{n+1} - T_{ij}^n}{\Delta t} = k\left(\frac{T_{i-1j}^{n+1} + T_{ij-1}^{n+1} - 4T_{ij}^{n+1} + T_{i+1j}^{n+1} + T_{ij+1}^{n+1}}{h^2}\right)$$

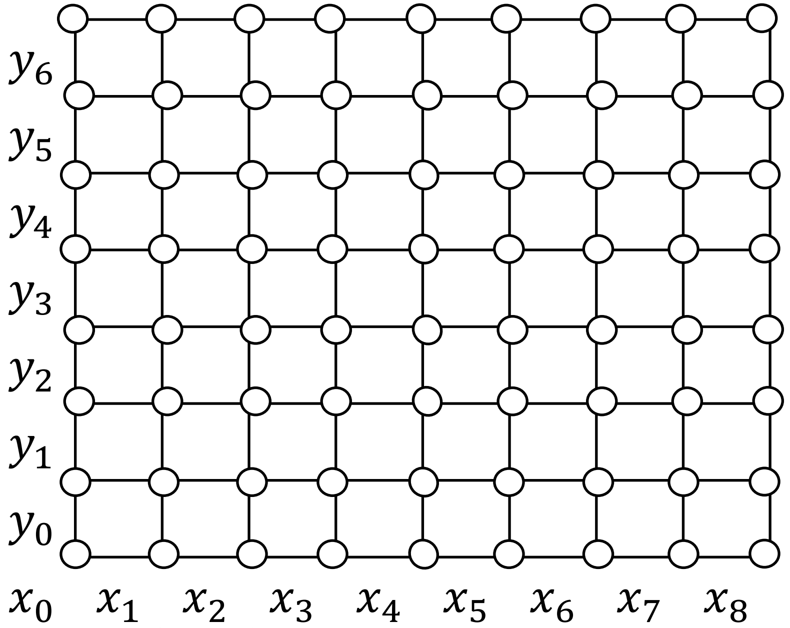
Backward Euler Scheme

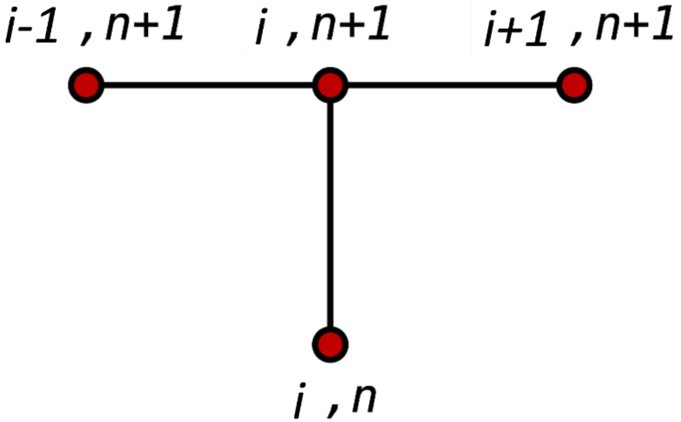
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\Delta x = \Delta y = h$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$

$$T_{ij}^{n+1} = T_{ij}^n + \frac{k\Delta t}{h^2}\left(T_{i-1j}^{n+1} + T_{ij-1}^{n+1} - 4T_{ij}^{n+1} + T_{i+1j}^{n+1} + T_{ij+1}^{n+1}\right)$$





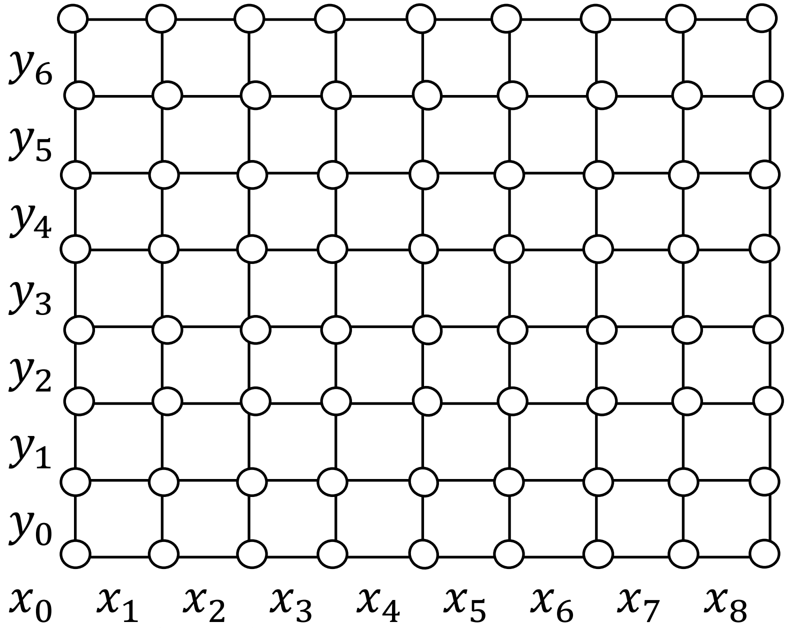
Backward Euler Scheme

$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

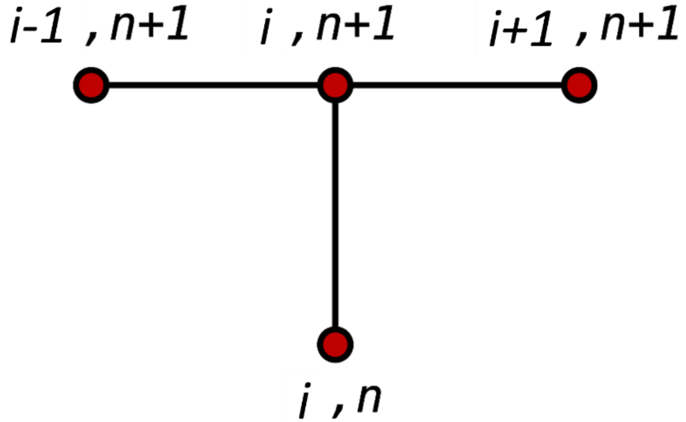
$$\Delta x = \Delta y = h$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$

$$F = \frac{k\Delta t}{h^2}$$



$$T_{ij}^{n+1} = T_{ij}^n + \frac{k\Delta t}{h^2}\left(T_{i-1j}^{n+1} + T_{ij-1}^{n+1} - 4T_{ij}^{n+1} + T_{i+1j}^{n+1} + T_{ij+1}^{n+1}\right)$$



$$T_{ij}^{n+1} = T_{ij}^n + F\left(T_{i-1j}^{n+1} + T_{ij-1}^{n+1} - 4T_{ij}^{n+1} + T_{i+1j}^{n+1} + T_{ij+1}^{n+1}\right)$$

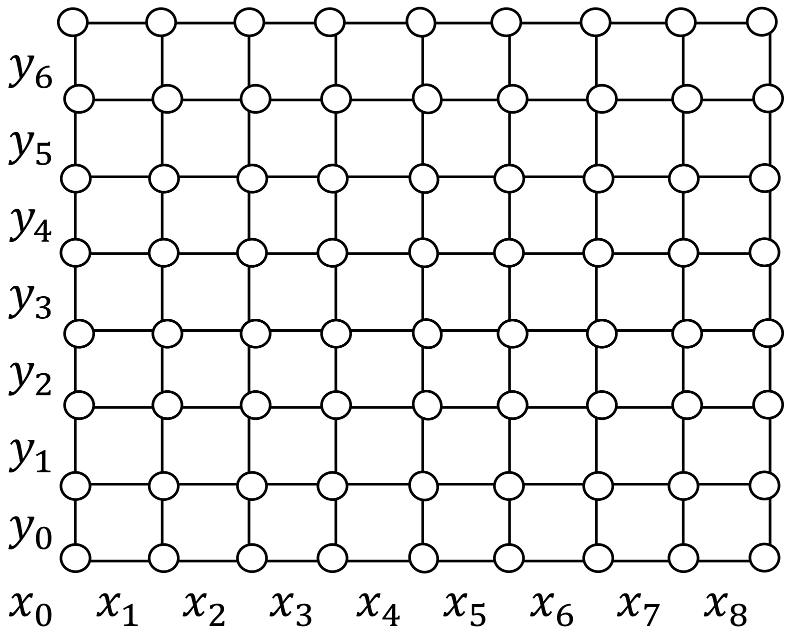
Backward Euler Scheme

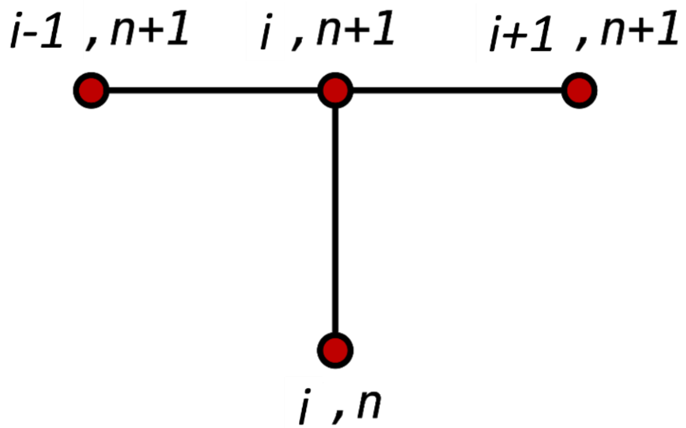
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\Delta x = \Delta y = h$$

$$T(x_i, y_j, t_n) = T_{ij}^n$$

$$F = \frac{k\Delta t}{h^2}$$





$$-FT_{i-1j}^{n+1} - FT_{ij-1}^{n+1} + (1+4F)T_{ij}^{n+1} - FT_{i+1j}^{n+1} - FT_{ij+1}^{n+1} = T_{ij}^n$$

Backward Euler Scheme

$$-FT_{i-1j}^{n+1} - FT_{ij-1}^{n+1} + (1+4F)T_{ij}^{n+1} - FT_{i+1j}^{n+1} - FT_{ij+1}^{n+1} = T_{ij}^{n}$$

$$(1,1): -FT_{01}^{n+1} - FT_{10}^{n+1} + (1+4F)T_{11}^{n+1} - FT_{21}^{n+1} - FT_{12}^{n+1} = T_{11}^{n}$$

$$(1,2): -FT_{02}^{n+1} - FT_{11}^{n+1} + (1+4F)T_{12}^{n+1} - FT_{22}^{n+1} - FT_{13}^{n+1} = T_{12}^{n}$$

$$(2,1): -FT_{11}^{n+1} - FT_{20}^{n+1} + (1+4F)T_{21}^{n+1} - FT_{31}^{n+1} - FT_{22}^{n+1} = T_{21}^{n}$$

$$(1,0): -FT_{00}^{n+1} + (1+4F)T_{10}^{n+1} - FT_{20}^{n+1} - FT_{11}^{n+1} = T_{10}^{n}$$

$$(0,1): -FT_{00}^{n+1} + (1+4F)T_{01}^{n+1} - FT_{11}^{n+1} - FT_{01}^{n+1} = T_{00}^{n}$$

Backward Euler Scheme

$$-FT_{i-1j}^{n+1} - FT_{ij-1}^{n+1} + (1+4F)T_{ij}^{n+1} - FT_{i+1j}^{n+1} - FT_{ij+1}^{n+1} = T_{ij}^{n}$$

$$(1,1): -FT_{01}^{n+1} - FT_{10}^{n+1} + (1+4F)T_{11}^{n+1} - FT_{21}^{n+1} - FT_{12}^{n+1} = T_{11}^{n}$$

$$(1,2): -FT_{02}^{n+1} - FT_{11}^{n+1} + (1+4F)T_{12}^{n+1} - FT_{22}^{n+1} - FT_{13}^{n+1} = T_{12}^{n}$$

$$(2,1): -FT_{11}^{n+1} - FT_{20}^{n+1} + (1+4F)T_{21}^{n+1} - FT_{31}^{n+1} - FT_{22}^{n+1} = T_{21}^{n}$$

$$\cancel{(1,0): -FT_{00}^{n+1} + (1+4F)T_{10}^{n+1} - FT_{20}^{n+1} - FT_{11}^{n+1} = T_{10}^{n}}$$

$$\cancel{(0,1): -FT_{00}^{n+1} + (1+4F)T_{01}^{n+1} - FT_{11}^{n+1} - FT_{01}^{n+1} = T_{00}^{n}}$$

Backward Euler Scheme

$$-FT_{i-1j}^{n+1} - FT_{ij-1}^{n+1} + (1+4F)T_{ij}^{n+1} - T_{i+1j}^{n+1} - T_{ij+1}^{n+1} = T_{ij}^{n}$$

$$(1,1): -FT_{01}^{n+1} - FT_{10}^{n+1} + (1+4F)T_{11}^{n+1} - FT_{21}^{n+1} - FT_{12}^{n+1} = T_{11}^{n}$$

$$(1,2): -FT_{02}^{n+1} - FT_{11}^{n+1} + (1+4F)T_{12}^{n+1} - FT_{22}^{n+1} - FT_{13}^{n+1} = T_{12}^{n}$$

$$(2,1): -FT_{11}^{n+1} - FT_{20}^{n+1} + (1+4F)T_{21}^{n+1} - FT_{31}^{n+1} - FT_{22}^{n+1} = T_{21}^{n}$$

$$A = \begin{bmatrix}
1 & 0 & 0 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
0 & 1 & 0 & \ddots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \vdots \\
\vdots & -F & 0 & \dots & -F & 1+4F & \dots & -F & \dots & \dots & -F & \dots & \vdots \\
\vdots & \vdots & -F & \ddots & \ddots & -F & 1+4F & \dots & -F & \dots & \dots & \dots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & -F & \ddots & \dots & \dots & \dots & \dots & \dots & \vdots \\
\vdots & \vdots & \dots & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \dots & \dots & \dots & 0 \\
0 & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 1 & \dots & \dots & \dots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 0 & \dots & \dots & \dots & 1
\end{bmatrix}
\quad
T^{n+1} = \begin{bmatrix}
T_{00}^{n+1} \\ T_{01}^{n+1} \\ \vdots \\ T_{0N_y-1}^{n+1} \\ T_{10}^{n+1} \\ T_{11}^{n+1} \\ \vdots \\ T_{1N_y-1}^{n+1} \\ \vdots \\ \vdots \\ T_{N_x-10}^{n+1} \\ T_{N_x-11}^{n+1} \\ \vdots \\ T_{N_x-1N_y-1}^{n+1}
\end{bmatrix}
\quad
b = \begin{bmatrix}
T_b \\ T_b \\ \vdots \\ T_b \\ T_l \\ T_{11}^{n} \\ \vdots \\ T_{1N_y-1}^{n} \\ \vdots \\ \vdots \\ T_u \\ T_u \\ \vdots \\ T_u
\end{bmatrix}$$

End of Lecture